ON A POSSIBLE FORMULATION OF PARTICLE DYNAMICS IN TERMS OF THERMODYNAMIC CONCEPTUALIZATIONS AND THE ROLE OF ENTROPY ON IT

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Monterey, California

THESIS

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THERMODYNAMIC CONCEPTUALIZATIONS
AND THE ROLE OF ENTROPY IN IT

by

Pharis Edward Williams

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TABLE OF CONTENTS

LIST OF SYMBOLS

H	classical Hamiltonian/thermodynamic enthalpy
L	classical Lagrangian
T	classical kinetic energy
V	classical potential energy
u	thermodynamic internal energy
S	thermodynamic entropy/general arc length
A	thermodynamic Helmholtz energy/classical action
G	thermodynamic Gibbs energy
S	mechanical entropy
F	velocity dependent force
F	velocity independent force
q	generalized coordinate
$\dot{q}$	generalized velocity
$\bar{d}W$	element of work
Q	non-mechanical energy transferred
U	energy of the system
C	energy capacity
ϕ	integrating factor for dQ
H	mechanical enthalpy/isentropic Hamiltonian
K	mechanical Helmholtz function
G	mechanical Gibbs function
A	mechanical free energy
L	Lagrangian of isentropic system
τ	kinetic entropy

$\mathcal{V}$	potential entropy
p	momentum
m	mass
E	total relativistic energy
$g_{\alpha\beta}$	coordinate metric elements
h_{ij}	general line element coefficients
α	Greek indices may take on values 1, 2, or 3
i	Latin indices may take on values 0, 1, 2, or 3
$\tilde{p}_i$	component of momentum in space-entropy manifold
$\tilde{\mathcal{L}}$	Lagrangian in space-entropy manifold/rate of 4-d arc length change
$\tilde{\mathcal{H}}$	Hamiltonian in space-entropy manifold
$\tilde{T}$	kinetic energy in space-entropy manifold
$\tilde{\Phi}$	a scalar potential, $\tilde{\Phi}(q^0,q^1,q^2,q^3,\dot{q}^0)$
$\tilde{A}_\alpha$	component of vector potential, $\tilde{A}_\alpha(q^0,q^1,q^2,q^3,\dot{q}^0)$
$\tilde{E}_\alpha,\tilde{B}_\alpha$	components of fields, $\tilde{E}_\alpha(q^0,q^1,q^2,q^3,\dot{q}^0)$
$\underset{\sim}{L}$	rate of 4-d arc length change with respect to change in entropy
$\underset{\sim}{\Phi}$	a scalar potential, $\underset{\sim}{\Phi}(q^0,q^1,q^2,q^3)$
A_α	component of vector potential, $\underset{\sim}{A}_\alpha(q^0,q^1,q^2,q^3)$
$\underset{\sim}{E}_\alpha,\underset{\sim}{B}_\alpha$	components of fields, $\underset{\sim}{E}_\alpha(q^0,q^1,q^2,q^3)$

I. INTRODUCTION

This study presents a new formulation of general dynamic systems. This formulation includes both thermodynamic and mechanistic concepts. It is shown that even relativistic mechanics with its characteristic occurrence of a limiting velocity can be described on the basis of thermodynamic concepts. This approach also sheds light on the role of entropy in the description of non-conservative mechanical systems.

Physical theories are proposed for many reasons. One of these might be to describe, or understand, a familiar phenomenon which had no prior description or explanation. Another might be the discovery of a new phenomenon, or the results of a new experiment, which has no explanation within the scope of existing theories. Still another is to bring the description of phenomena which at first appear to be unrelated together under a unifying theory. The motivation, or objective, involved in the development and proposal of any theory plays an important role in the philosophical basis upon which the theory is developed and, therefore, may become a part of the theory itself.

The motivation for this investigation arose from a number of questions which one could rather naively ask:

The first concerns the requirement of Lorentz covariance of all laws of nature. Ample theoretical and experimental evidence exists for this requirement when electromagnetic

forces are considered. Electromagnetic waves are accurately described by Maxwell's equations and propagate in free space with the speed c for every inertial observer. But what forces us to require Lorentz covariance for all laws of nature, even those dealing with other than the electromagnetic interactions? What about the gravitational or the weak interaction? Gravitational waves have been predicted and their detection has recently been claimed. Could these not travel in free space with some other velocity b? Are there any other reasons than aesthetics or the principle of Occams razor, which asks us to consider only the simplest system of laws, that there is only one characteristic velocity in nature?

A second question concerns the role of time assymmetry. The equations of motion in both Newtonian and relativistic mechanics are time symmetrical. Yet nature displays a directivity that would not be described by the universal application of time symmetrical laws. The most vivid display of this directivity in nature is in thermodynamics where the principle of increasing thermodynamic entropy has many uses. Then should not all dynamics share such a directivity? If so, this directivity would not be seen in time symmetrical laws of motion.

Dynamics, as described by relativity, has a limiting value of velocity which is the speed of light. This notion of a limiting value also appears in thermodynamics in the absolute zero temperature. This similarity between thermodynamics and relativistic dynamics, the desire to introduce

the possibility of temporal directivity into mechanics, and the strength and generality of the basic laws of thermodynamics focused this investigation.

The objective of this investigation was to determine whether or not the logical structure of classical thermodynamics could yield dynamical laws which could be applied to mechanical systems and produce equations of motion which would contain existing dynamical theories as limiting, or special cases, and provide the directivity seen in nature. The following proposed formulation of a dynamical theory is the result of such an investigation. It should perhaps be stated here that this formulation does not represent an attempt to base mechanical dynamics upon thermodynamics itself but to use the logical formulism of thermodynamics as a common basis for different branches of dynamics.

The investigation is based upon the formulation of three dynamical laws identical in structure to the three laws of thermodynamics. The only difference between the development presented here up to, and including, stability conditions and the development of thermodynamics is that velocity, position, and force will be used as thermodynamic variables instead of temperature, volume and pressure.

It may seem that little is to be gained by simply rewriting thermodynamics and particle dynamics in this fashion. However if the possibility exists for thermodynamics and particle dynamics to result from the application of the same laws,

these laws must be identical with thermodynamic laws when thermodynamic variables are used. To see the results of these laws applied to particle dynamics requires the use of the variables normally used for particle motion description.

Section II presents the three proposed dynamical laws. The section also includes the axiomatic development of the dynamical second law and determines an integrating factor which makes the differential energy exchange between the system and the environment a perfect differential. The integrating factor is shown to be a function only of the velocity. An argument, following Caratheodory's, proves the existence of a unique limiting velocity. The concept of mechanical entropy is introduced and the principle of increasing mechanical entropy is presented.

In thermodynamics other state functions are defined and prove very useful in different applications. The same state functions for the mechanical system should also play similar useful roles. Section III defines these state functions and derives the mechanical Maxwell relations based on these functions.

Section IV derives the equilibrium and stability conditions based on the mechanical state functions. The analysis results in quadratic forms in various variables which express the stability conditions. These are the required quadratic forms.

Up to this point thermodynamic logic has been used exclusively. But the development has shown the existence of a

limiting velocity and the existence of the mechanical entropy. The development also displays the "natural" variables of particle dynamics. These variables may be seen in the quadratic forms which express the stability conditions. This is the point where this thesis introduces a new idea.

The new idea is the adoption of the quadratic forms and variables that express the stability conditions as the metrics and natural variables which govern particle dynamics. The natural variables that appear in the simplest quadratic form are the space coordinates and the mechanical entropy. The metric in relativistic particle dynamics is a metric involving space and time as the variables. Therefore Section V deviates from a logical abstract approach, which suggests the investigation continue on by adopting the metric, by looking at the most general motion, and showing that in special cases the allowed motion is identical with the motion of Newtonian or relativistic dynamics. This digression demonstrates the consistency between the proposed thermodynamic description of a mechanical isentropic system and the description provided by Newtonian and relativistic mechanics. It provides a measure of confidence in the abstract approach which is picked up again in Section VI. Here the arc element and parameterization are discussed and the resulting equations of motion are presented.

Appendices A, B, and C provide proofs and developments in support of the text. Appendix E briefly discusses a

possible prediction of expanding planetary orbits. Consistency with relativistic particle dynamics is again addressed in Appendix D where the space-time manifold is shown to be the result from the application of the principle of increasing mechanical entropy to the space-entropy manifold. This represents the completion of the logical progression which formulates the dynamical laws, deriving the quadratic forms (which are taken as the metric), applies the dynamical second law in the form of the mechanical entropy principle, and shows that the resulting manifold is a space-time manifold which, for the special case of a Euclidean manifold, is the Minkowski space of special relativity. Appendix F presents a brief look at the equations of motion which result from two different methods of parameterizing the space-entropy manifold.

Figure 1 is a flow chart which indicates the logical structure of the text and the manner in which the Appendices fit into this structure.

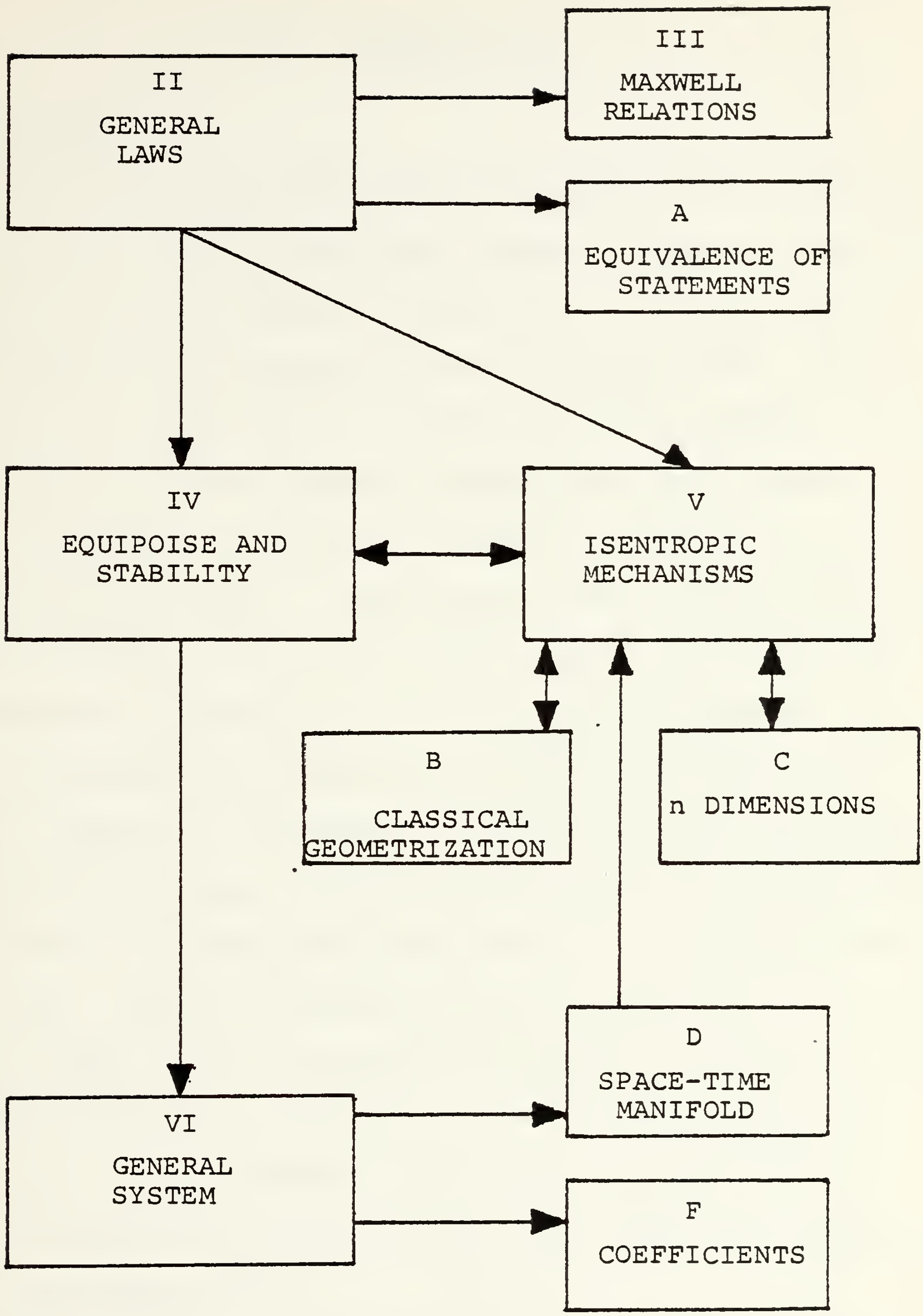
II
GENERAL LAWS
III
MAXWELL RELATIONS
A
EQUIVALENCE OF STATEMENTS
IV
EQUIPOISE AND STABILITY
V
ISENTROPIC MECHANISMS
B
CLASSICAL GEOMETRIZATION
C
n DIMENSIONS
D
SPACE-TIME MANIFOLD
VI
GENERAL SYSTEM
F
COEFFICIENTS

Figure 1

II. GENERAL LAWS

A. CONCEPTS

In the following development physical concepts are necessary, as are symbols for these concepts. Because this development will merge certain thermodynamic conceptualizations into mechanics a notational dilemma must be faced. At the one hand it is desired to preserve the thermodynamic conceptualization by using familiar symbols from that theory. On the other hand it is really mechanical systems for which a description is sought. The formalism then looks either like thermodynamics with familiar thermodynamic quantities replaced by mechanical quantities, or it looks like mechanics into which thermodynamic quantities introded. In either case there is danger of confusion. One could evade the dilemma by choosing entirely different symbols for the variables of the theory. But then the whole takes an artificially abstract character. Since the purpose of this formulation is to bring out the power of the thermodynamic conceptualization it was decided to use the suggestiveness of the thermodynamic or mechanical symbols whenever convenient and the reader is asked to keep an open mind and not make premature associations with the symbols used.

As an example, in mechanics there are energy concepts such as "energy of the system", Hamiltonian, Lagrangian, kinetic energy, etc. In classical thermodynamics there is

"internal energy of the system", entropy, enthalpy, and free energy. If a dynamical theory for mechanical systems is developed from the logic of thermodynamics the various thermodynamic energy concepts may be expected to have analogies applicable in the mechanical system. Thus there are three types of theoretical energy concepts, with their associated symbols, involved;

i. mechanical energy concepts (symbols H, L, T, V, etc.),

ii. thermodynamic energy concepts (symbols u, S, H, A, G, etc.), and

iii. mechanical concepts analogous to thermodynamic concepts developed here (symbols to be chosen).

Difficulty may arise with the choice of symbols and the words used to denote concepts. In particular consider the thermodynamic "internal energy of the system." The mechanical analoque of this concept will be called "energy of the system." It is natural to equate this in one's mind with the Hamiltonian in classical or relativistic mechanics, however it will be seen that this association is not appropriate in general. Symbols used are chosen in two ways. One method is to use a symbol which identifies the concept with its origin in the thermodynamic logic, thus the script $\mathcal{S}$ identifies the mechanical entropy concept with its analoque, the thermodynamic entropy S. The other method is to use the symbol with its mechanical role in mind, here the script $\mathcal{F}$ will be used to denote the concept of a force, however this

force is not the same as the force, denoted by F, in existing mechanics theories.

The reader may also be tempted to seek an immediate interpretation of a new idea rather than following along with the development of the abstract logic to a more appropriate point to make an interpretation between the abstract concept and physical reality. A reader with a strong background in relativity may be thinking in terms of inertial reference frames and transformations while others may be thinking of reversibility and irreversibility. Transformations between inertial reference frames are not considered in this investigation. Neither is extensive investigation into reversibility/irreversibility attempted. Therefore the reader is cautioned not to prematurely apply his knowledge of another theory in interpreting a concept presented here.

The following list of definitions presents some of the concepts which will be used in later developments. Just as the useful but non-operational definition of heat, which is "Heat is that which is transferred between a system and its surroundings as a result of temperature differences only," seems vague when first seen some of the following may not be immediately clear. Later developments and use of the concept should help to clarify the definition:

a. A "dynamic system" may be any physical system of any number of parameters, where parameters are considered to include constants (constants of the motion, integration constants, and/or universal constants, i.e. gravitational

constant) and dynamic parameters.

b. "Dynamic parameters" are quantities necessary to describe the system and include variables such as velocity $\dot{q}$, where $\dot{q} \equiv dq/dt$, and position q as well as parameters such as time, which is thought of here as a parameter of the motion rather than a coordinate as in relativity theories.

c. A "state" is specified by a set of values of all the parameters necessary for the description of the system.

d. "Equipoise" prevails when the state of the system does not change in time. The word "equilibrium" could have been used here except its use in connection with a mechanical system will tend to cause the reader to think in terms of an existing dynamic law, such as Newton's second law, which makes the definition of mechanical equilibrium more readily understood. This investigation seeks a dynamic formulation therefore care must be taken here to avoid premature interpretation. The meaning of equipoise will become clearer after the conditions for equipoise are discussed and the dynamical laws are formulated for then the relationship between equipoise and classical mechanical equilibrium may be seen.

e. The "equation of state" is a functional relationship among the dynamic parameters for a system. If F, $\dot{q}$, and q are the dynamic parameters of the system, an equation of state may take the form

$$f(F,\dot{q},q) = 0$$

which reduces the number of independent variables of the system from three to two, f must be continuous and at least twice differentiable. It is useful to represent such a system by a point in the three-dimensional $(F,\dot{q},q)$ space as shown in Figure 2.

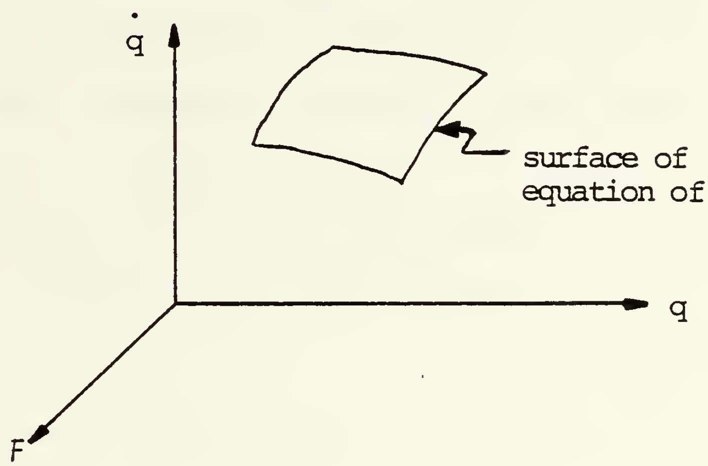

Figure 2. Geometrical representation of the equation of state

The equation of state then denotes a surface in this space. Any point lying on this surface represents a possible kinematic state of the system.

If the system is represented by the nine variables F_1, F_2, F_3, $\dot{q}^1$, $\dot{q}^2$, $\dot{q}^3$, q^1, q^2, q^3, then the equation of state defines a hyper-surface in this nine-dimensional space.

f. A "dynamic transformation" is a change of state. If the initial state is an equilibrium state, the transformation can be brought about only by changes in the external constraints, such as forces, on the system. The transformation is quasi-static if the constraints change so slowly

that at any moment the system is approximately in equilibrium. It is "reversible" if the transformation retraces its history in time when the constraints retrace their history in time. A reversible transformation is quasi-static, but the converse is not necessarily true. As an example, in thermodynamics a gas that freely expands into successive infinitessimal volume elements undergoes a quasi-static transformation but not a reversible one.

g. The concept of work is the same as in mechanics in that, when an equation of state exists,

$$\bar{d}\mathcal{W} \equiv \mathcal{F}_i \, dq^i$$

where the summation convention $\mathcal{F}_i \, dq^i = \sum_i \mathcal{F}_i \, dq^i$ is used. The use of the script $\mathcal{F}$ will be to indicate that this variable, "force", is considered to be a function of the position and velocity.

h. Numerous energy concepts arise in later developments and care will be required in notation to minimize confusion. Therefore, all energy functions introduced in this dynamic theory will be denoted by script letters while capital letters will be used for energy functions in other theories. Non-mechanical energy absorbed by the system will be denoted by Q. If the system is a thermodynamic system Q is the heat while for an electromagnetic system Q may be radiant energy absorbed by the system.

i. A "reservoir" is a system so large that the gain or loss of any finite amount of energy does not change its state.

j. A system is "isolated" if no non-mechanical energy, Q, is exchanged between it and the external universe. Any transformation the system can undergo in isolation is called a "Q-conservative process". The word "Q-conservative" is used to emphasize that this definition of a conservative process is more general than the definition of a conservative system in classical mechanics. The distinction between a Q-conservative system and the classical conservative system may be seen later.

The energy of the system, which represents the energy possessed by the system, is considered to be

$$\mathcal{U}(q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n, c_1, \ldots, c_m)$$

$d\mathcal{U}$ will be assumed to be a perfect differential. The reader is again reminded that this function can not in general be equated to the Hamiltonian of classical or relativistic mechanics. The relationship between the function $\mathcal{U}$ and the energy concepts of classical mechanics can be seen after equations of motion have been formulated. Therefore to minimize the possibility of confusion the function $\mathcal{U}$ will be referred to as the "system energy."

It will be supposed that functions defining various kinds of energies depend on a number of parameters. The totality

of the parameters, both dynamic parameters and constants, need not be unique, but whatever particular choice of a set of dynamic parameters (variables) is made, it shall be assumed that they are independent. In some situations the variables may be determined as functions of a scalar t (usually time) so that one can regard them as defining a curve C; $x^i = x^i(t)$, characterizing a certain process where the x^i indicate the set of independent variables.

B. FIRST LAW

The concept of conservation of energy is fundamental to all branches of physics and therefore represents a logical beginning for a generalized theory. Therefore, in terms of generalized coordinates the notion of work, or mechanical energy, is considered linear forms of the type

$$\bar{d}W = F_i(q^1,\ldots,q^n,\dot{q}^1\ldots,\dot{q}^n,c_1,\ldots,c_m)dq^i; \quad (i=1,2,\ldots,n)$$

The line integral $\int_C F_i dq^i$ then represents the work done along the path C by the generalized forces.

A system may acquire energy other than mechanical, such energy acquisition is denoted $\bar{d}Q$.

The energy of the system, which represents the energy possessed by the system, is considered to be

$$U(q^1,\ldots,q^n,\dot{q}^1,\ldots,\dot{q}^n,c_1,\ldots,c_m).$$

DU will be assumed to be a perfect differential.

With these concepts then the generalized law of conservation of energy has the form

$$\bar{d}Q \equiv dU - \bar{d}W$$

$$= dU - F_\alpha dq^\alpha \; ; \qquad (\alpha=1,2,3) \qquad \text{(II-1)}$$

$$= \frac{\partial U}{\partial \dot{q}^\alpha} d\dot{q}^\alpha - [F_\alpha(\dot{q},q) + F_\alpha]dq^\alpha$$

where

$$F_\alpha(\dot{q},q) \equiv \frac{\partial U}{\partial \dot{q}^\alpha} \, .$$

Positive $\bar{d}Q$ is taken as energy added to the system by means other than mechanical and F_α is taken as the component of the generalized force acting on the system. Some systems may not involve forces $F_\alpha(\dot{q},q)$, which are functions only of position in classical mechanics. It will be seen that when they do exist the forces F_α, in the conservation of energy statement play a role analogous to reactive forces. It will also be seen that if there are no forces $F_\alpha(\dot{q},q)$, the role of the forces F_α is somewhat different from a role analogous to reactive forces.

In an infinitesimal transformation, the first law is equivalent to the statement that the differential

$$dU = \bar{d}Q + F_\alpha dq^\alpha$$

is exact. That is, there exists a function U whose differential is dU; or the integral $\int dU$ is independent of the path of the integration and depends only on the limits of integration. This condition is not shared by $\bar{d}Q$ or $\bar{d}W$.

As an example a one-dimensional case with the variables F, $\dot{q}$, and q will be considered.

Given a differential of the form $df = g(A,B)dA + h(A,B)dB$, the condition that df is exact is $\frac{\partial g}{\partial B} = \frac{\partial h}{\partial A}$. To explore some of the consequences of the exactness of dU consider a system whose parameters are F, $\dot{q}$, q. Any pair of these three parameters may be chosen to be the independent variables that completely specify the state of the system. The other parameter is then determined by the equation of state, for example, consider $U = U(F,q)$, then

$$dU = \left(\frac{\partial U}{\partial F}\right)_q dF + \left(\frac{\partial U}{\partial q}\right)_F dq ,$$

the requirement that dU be exact immediately leads to the result

$$\frac{\partial}{\partial q}\left[\left(\frac{\partial U}{\partial F}\right)_q\right]_F = \frac{\partial}{\partial F}\left[\left(\frac{\partial U}{\partial q}\right)_F\right]_q .$$

The following equations, expressing the energy absorbed by a system during an infinitesimal reversible transformation

are easily obtained by successively choosing as independent variables the pairs (F,q), $(F,\dot{q})$ and $(\dot{q},q)$;

$$đQ = \left(\frac{\partial U}{\partial F}\right)_q dF + \left[\left(\frac{\partial U}{\partial q}\right)_F - F\right] dq,$$

$$đQ = \left[\left(\frac{\partial U}{\partial \dot{q}}\right)_F - F\left(\frac{\partial q}{\partial \dot{q}}\right)_F\right] d\dot{q} + \left[\left(\frac{\partial U}{\partial F}\right)_{\dot{q}} - F\left(\frac{\partial q}{\partial F}\right)_{\dot{q}}\right] dF,$$

$$đQ = \left(\frac{\partial U}{\partial \dot{q}}\right)_q d\dot{q} + \left[\left(\frac{\partial U}{\partial q}\right)_{\dot{q}} - F\right] dq.$$

These equations are of little practical use in their present form because the partial derivatives that appear are unknown. However, from these equations the "energy capacities" may be defined as

$$C \equiv \frac{\Delta Q}{\Delta \dot{q}},$$

then from the above $đQ$ equations the energy capacities are seen to be

$$C_q \equiv \left(\frac{\Delta Q}{\Delta \dot{q}}\right)_q = \left(\frac{\partial U}{\partial \dot{q}}\right)_q$$

and

$$C_F \equiv \left(\frac{\Delta Q}{\Delta \dot{q}}\right)_F = \left(\frac{\partial U}{\partial \dot{q}}\right) - F\left(\frac{\partial q}{\partial \dot{q}}\right)_F .$$

C. SECOND LAW

1. Transformation Statement

There are processes which satisfy the first law but which are not observed in nature. The purpose of the dynamical second law is to incorporate such experimental facts into the model of dynamics.

The logic of thermodynamics offers two approaches to a generalized second law. The first approach consists of two equivalent generalized statements the first of which, for the mechanical system, may be stated as;

I. there exists no dynamic process whose sole effect is to extract a quantity of energy from a given reservoir (or source) and to convert it entirely into work.

The second statement is given in Appendix A and is shown in the Appendix to be equivalent to the first statement.

2. Axiomatic Statement

A second approach to the dynamical second law has been provided by the Greek mathematician Caratheodory, who presented an axiomatic development of the second law of thermodynamics. This development is presented here with the notion of a mechanical system in mind rather than a thermodynamic system to demonstrate the applicability of the logic to any type of system.

a. Existence of Constant Energy Surfaces.

In the statement of the first law $d\mathcal{U}$ is considered as a perfect differential; however $\bar{d}Q$ and $\mathcal{F}dq$ are not, in general, perfect differentials. Therefore consider a process

in which the system exchanges energy with its surroundings; then an axiom analogous to Caratheodory's axiom may be cited;

Axiom: In the neighborhood (however close) of any equipoise state of a system of any number of dynamic coordinates, there exist states that cannot be reached by reversible Q-conservative ($\bar{d}Q = 0$) processes.

When the variables are thermodynamic variables the Q-conservative processes are known as adiabatic processes.

A reversible process is one that is performed in such a way that, at the conclusion of the process, both the system and the local surroundings may be restored to their initial states, without producing any change in the rest of the universe.

Consider a system whose independent coordinates are a generalized displacement denoted q, a generalized velocity $\dot{q}$ (with $\dot{q} \equiv dq/dt$), and a generalized force F. It will be shown that the Q-conservative curve comprising all equipoise states accessible from the initial state, i, may be expressed by

$$\sigma = \sigma(\dot{q},q) = \text{constant},$$

where σ represents some as yet undetermined function. Curves corresponding to other initial states would be represented by different values of the constant.

Reversible Q-conservative curves cannot intersect, for if they did it would be possible, as shown in Figure 3, to proceed from an initial equipoise state i, at the point of intersection, to two different final states f_1 and f_2, having the same q, along reversible Q-conservative paths, which is not allowed by the axiom.

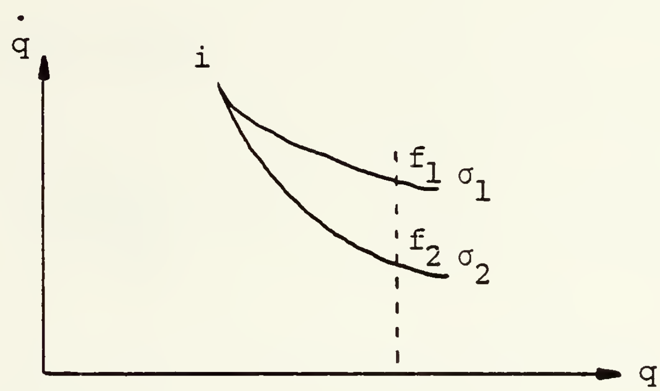

Figure 3. If two reversible Q-conservative curves could intersect, it would be possible to violate the axiom by performing the cycle $if_1 f_2i$.

b. Integrability of $đQ$

When the system can be described with only two independent variables, such as on the Q-conservative curve, then if these variables are $\dot{q}$ and q, and F is a generalized force,

$$đQ = dU - Fdq \ . \qquad \text{(II-1A)}$$

Regarding $U = U(\dot{q},q)$ then

$$\bar{d}Q = \left(\frac{\partial \mathcal{U}}{\partial \dot{q}}\right)_q d\dot{q} + \left[\left(\frac{\partial \mathcal{U}}{\partial q}\right)_{\dot{q}} - F\right] dq, \tag{II-2}$$

where $\left(\frac{\partial \mathcal{U}}{\partial \dot{q}}\right)_q$, F, and $\left(\frac{\partial \mathcal{U}}{\partial q}\right)_{\dot{q}}$ are functions of $\dot{q}$ and q.

A Q-conservative process for this system is

$$\left(\frac{\partial \mathcal{U}}{\partial \dot{q}}\right)_q d\dot{q} + \left[\left(\frac{\partial \mathcal{U}}{\partial q}\right)_{\dot{q}} - F\right] dq = 0 \ . \tag{II-3}$$

Solving for $d\dot{q}/dq$ yields

$$\frac{d\dot{q}}{dq} = \frac{-\left[\left(\frac{\partial \mathcal{U}}{\partial q}\right)_{\dot{q}} - F\right]}{\left(\frac{\partial \mathcal{U}}{\partial \dot{q}}\right)_q} \ .$$

The right hand member is a function of $\dot{q}$ and q, and therefore the derivative $d\dot{q}/dq$, representing the slope of a Q-conservative curve on a $(\dot{q},q)$ diagram, is known at all points. Equation (II-3) has therefore a solution consisting of a family of curves, see Figure 4, and the curve through any one point may be written

$$\sigma = \sigma(\dot{q},q) = \text{constant.}$$

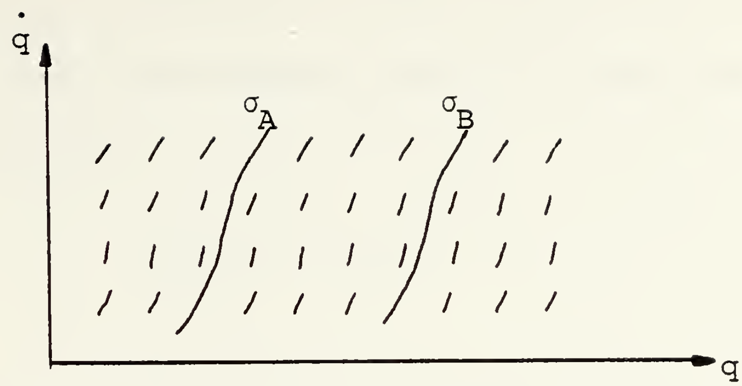

Figure 4. The first law, through equation (II-3) fills the $(\dot{q},q)$ space with slopes specified at each point. The σ curves represent the solution curves whose tangents are the required slopes. The second law requires that these curves do not intersect.

A set of curves is obtained when different values are assigned to the constant. The existence of the family of curves $\sigma(\dot{q},q)$ = constant, generated by equation (II-3), representing reversible Q-conservative processes, follows from the fact that there are only two independent variables and not from any law of physics. Thus it can be seen that the first law may be satisfied by any of these σ = constant curves. The axiom requires that these curves do not intersect. Therefore the axiom, together with the first law, leads to the conclusion that: through any arbitrary initial-state point, all reversible Q-conservative processes lie on a curve, and Q-conservative curves through other initial states determine a family of non-intersecting curves.

To see the results of this conclusion consider a system whose coordinates are the generalized velocity $\dot{q}$, the generalized displacement q and the generalized force F. The first law is

$$\bar{d}Q = d\mathcal{U} - \mathcal{F}dq \qquad \text{(II-4)}$$

where $\mathcal{U}$ and $\mathcal{F}$ are functions of $\dot{q}$ and q. Since the $(\dot{q},q)$ surface is subdivided into a family of non-intersecting Q-conservative curves

$$\sigma(\dot{q},q) = \text{constant}$$

where the constant can take on various values σ_1, σ_2, ... any point in the surface may be determined by specifying the value of σ along with q so that $\mathcal{U}$, as well as $\mathcal{F}$, may be regarded as functions of σ and q. Then

$$d\mathcal{U} = \left(\frac{\partial \mathcal{U}}{\partial \sigma}\right)_q d\sigma + \left(\frac{\partial \mathcal{U}}{\partial q}\right)_\sigma dq \qquad \text{(II-5)}$$

and

$$\bar{d}Q = \left(\frac{\partial \mathcal{U}}{\partial \sigma}\right)_q d\sigma + \left[\left(\frac{\partial \mathcal{U}}{\partial q}\right)_\sigma - \mathcal{F}\right] dq$$

Since σ and q are independent variables this equation must be true for all values of $d\sigma$ and dq.

Suppose $d\sigma = 0$ and $dq \neq 0$. The provision that $d\sigma = 0$ is the provision for a Q-conservative process in which $\bar{d}Q = 0$. Therefore, the coefficient of dq must vanish. Then, in order for σ and q to be independent and for $\bar{d}Q$ to be zero when $d\sigma$ is zero, the equation for $\bar{d}Q$ must reduce to

$$\bar{d}Q = \left(\frac{\partial U}{\partial \sigma}\right)_q d\sigma,$$

with

$$\left(\frac{\partial U}{\partial q}\right)_\sigma = F.$$

Defining a function λ by

$$\lambda \equiv \left(\frac{\partial U}{\partial \sigma}\right)_q ,$$

then

$$\bar{d}Q = \lambda \, d\sigma , \tag{II-6}$$

where

$$\lambda = \lambda(\sigma,q) .$$

Now, in general, an infinitesimal of the type

$$Pdx + Qdy + Rdz + \dots ,$$

known as a linear differential form, or a Pfaffian expression, when it involves three or more independent variables, does not admit of an integrating factor. It is only because of the existence of the axiom that the differential form for $\bar{d}Q$ referring to a physical system of any number of independent coordinates possess an integrating factor.

Two infinitesimally neighboring reversible Q-conservative curves are shown in Figure 5. One curve is characterized by a constant value of the function σ_A, and the other by a slightly different value $\sigma_A + d\sigma = \sigma_B$. In any process represented by a displacement along either of the two Q-conservative curves $\bar{d}Q = 0$. When a reversible process connects the two Q-conservative curves energy $\bar{d}Q = \lambda d\sigma$ is transferred.

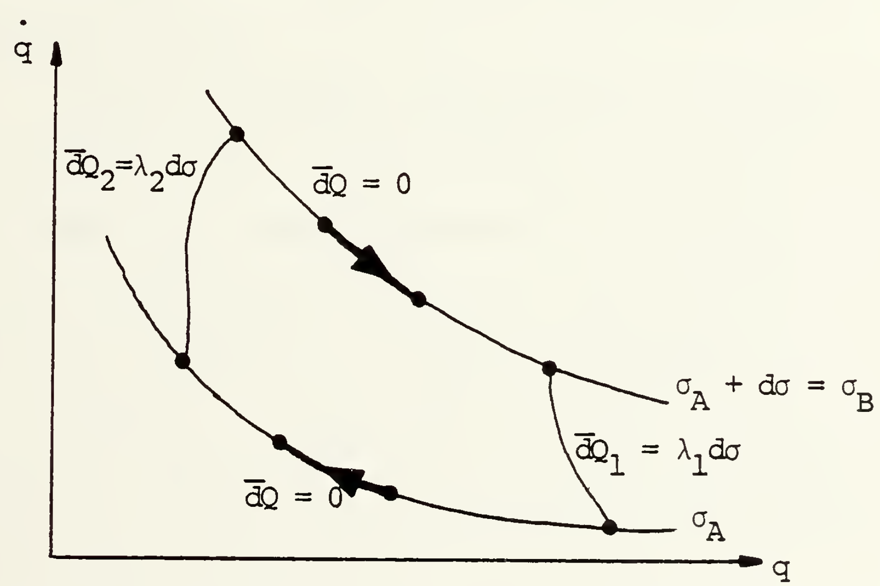

Figure 5. Two reversible Q-conservative curves, infinitesimally close, when the process is represented by a curve connecting the Q-conservative curves, energy $\bar{d}Q = \lambda d\sigma$ is transferred.

The various infinitesimal processes that may be chosen to connect the two neighboring reversible Q-conservative curves, shown in Figure 5, involve the same change of σ but take place at different λ. In general λ is a function of $\dot{q}$ and q. However it is obvious that λ may be expressed as a function of σ and $\dot{q}$. To find the velocity dependence of λ consider two systems, one and two, such that in the first system there are two independent coordinates $\dot{q}$ and q and the Q-conservative curves are specified by different values of the function σ of $\dot{q}$ and q. When energy $\bar{d}Q$ is transferred, σ changes by $d\sigma$ and $\bar{d}Q = \lambda d\sigma$ where λ is a function of σ and $\dot{q}$.

The second system has two independent coordinates $\dot{q}$, and $\hat{q}$ and the Q-conservative curves are specified by different values of the function $\hat{\sigma}$ of $\dot{q}$ and $\hat{q}$. When $\bar{d}\hat{Q}$ is transferred, $\hat{\sigma}$ changes by $d\hat{\sigma}$ and $\bar{d}\hat{Q} = \hat{\lambda} d\hat{\sigma}$ where $\hat{\lambda}$ is a function of $\hat{\sigma}$ and $\dot{q}$.

The two systems are related through the coordinate $\dot{q}$ in that both systems make up a composite system in which there are three independent coordinates $\dot{q}$, q, and $\hat{q}$ and the Q-conservative curves are specified by different values of the function σ_c of these independent variables.

Since $\sigma = \sigma(\dot{q},q)$ and $\hat{\sigma} = \hat{\sigma}(\dot{q},q)$, using the equations for σ and $\hat{\sigma}$, σ_c may be regarded as a function of $\dot{q}$, σ and $\hat{\sigma}$.

For an infinitesimal process between two neighboring Q-conservative surfaces specified by σ_c and $\sigma_c + d\sigma_c$, the energy transferred is $\bar{d}Q_c = \lambda_c d\sigma_c$ where λ_c is also a function of $\dot{q}$, σ, and $\hat{\sigma}$. Then

$$d\sigma_c = \frac{\partial\sigma_c}{\partial\dot{q}}\, d\dot{q} + \frac{\partial\sigma_c}{\partial\sigma}\, d \quad + \frac{\partial\sigma_c}{\partial\hat{\sigma}}\, d\hat{\sigma} \qquad \text{(II-7)}$$

Now suppose that in a process there is a transfer of energy $\bar{d}Q_c$ between the composite system and an external reservoir with energies $\bar{d}Q$ and $\bar{d}\hat{Q}$ being transferred, respectively, to the first and second systems, then

$$\bar{d}Q_c = \bar{d}Q + \bar{d}\hat{Q}$$

and

$$\lambda_c\, d\sigma_c = \lambda d\sigma + \hat{\lambda} d\hat{\sigma}\ ,$$

or

$$d\sigma_c = \frac{\lambda}{\lambda_c}\, d\sigma + \frac{\hat{\lambda}}{\lambda_c}\, d\hat{\sigma}\ . \qquad \text{(II-8)}$$

Comparing equations (II-7) and (II-8) for $d\sigma_c$ then

$$\frac{\partial\sigma_c}{\partial\dot{q}} = 0.$$

Therefore σ_c does not depend on $\dot{q}$, but only on σ and $\hat{\sigma}$. That is

$$\sigma_c = \sigma_c(\sigma,\hat{\sigma}) \ .$$

Again comparing the two expressions for $d\sigma_c$

$$\frac{\lambda}{\lambda_c} = \frac{d\sigma_c}{d\sigma} \quad \text{and} \quad \frac{\hat{\lambda}}{\lambda_c} = \frac{d\sigma_c}{d\hat{\sigma}} \ ,$$

therefore the two ratios λ/λ_c and $\hat{\lambda}/\lambda_c$ are also independent of $\dot{q}$, q and $\hat{q}$. These two ratios depend only on the σ's, but each separate λ must depend on the velocity as well (for example, if λ depended only on σ and on nothing else, the $\bar{d}Q = \lambda d\sigma$ would equal $f(\sigma)d\sigma$ which is an exact differential). In order for each λ to depend on the velocity and at the same time for the ratios of the λ's to depend only on the σ's, the λ's must have the following structure:

$$\lambda = \phi(\dot{q}) \ f(\sigma) \ ,$$

$$\hat{\lambda} = \phi(\dot{q}) \ \hat{f}(\hat{\sigma}) \ , \qquad \text{(II-9)}$$

and

$$\lambda_c = \phi(\dot{q}) \ g(\sigma,\hat{\sigma}) .$$

(The quantity λ cannot contain q, nor can $\hat{\lambda}$ contain $\hat{q}$, since λ/λ_c and $\hat{\lambda}/\lambda_c$ must be functions of the σ's only.)

Referring now only to the first system as representative of any system of any number of independent coordinates, the transferred energy is, from equations (II-9),

$$\bar{d}Q = \phi(\dot{q})\ f(\sigma)\ d\sigma \qquad \text{(II-10)}$$

Since $f(\sigma)d\sigma$ is an exact differential, the quantity $1/\phi(\dot{q})$ is an integrating factor for $\bar{d}Q$. It is an extraordinary circumstance that not only does an integrating factor exist for the $\bar{d}Q$ of any system, but this integrating factor is a function of velocity only and is the same function for all systems.

The fact that a system of two independent variables has a $\bar{d}Q$ which always admits an integrating factor regardless of the axiom is interesting, but its importance in physics is not established until it is shown that the integrating factor is a function of velocity only and that it is the same function for all systems.

c. The Absolute Velocity

The universal character of $\phi(\dot{q})$ makes it possible to define an absolute velocity. Consider a system of two independent variables $\dot{q}$ and q, for which two constant velocity curves and Q-conservative curves are shown in Figure 6. Suppose there is a constant velocity transfer of energy Q between the system and an external reservoir at the velocity $\dot{q}$, from a state b, on a Q-conservative curve characterized by the

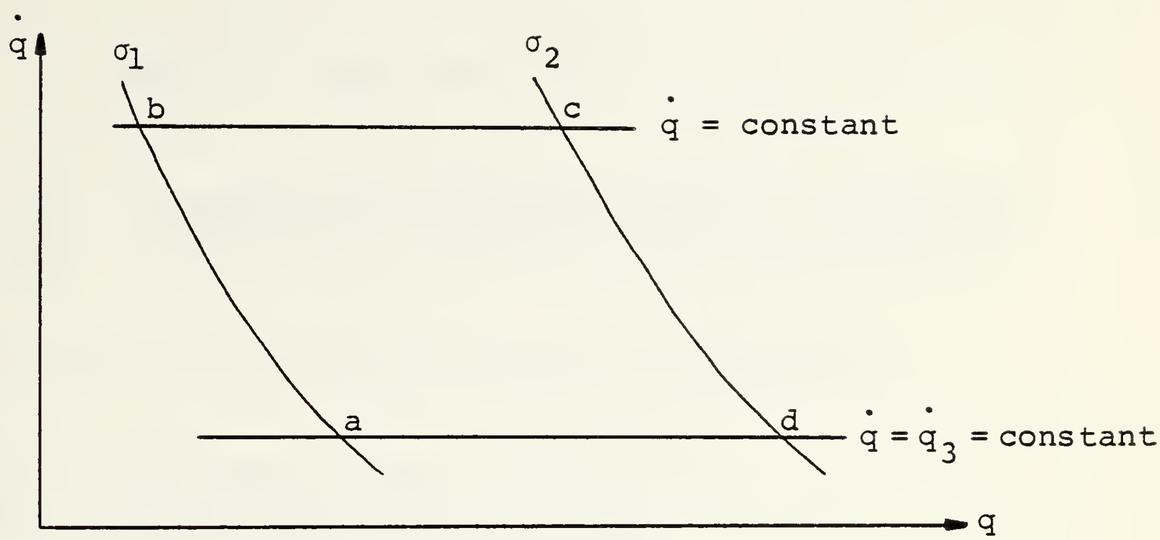

Figure 6. Two constant velocity energy transfers, Q at $\dot{q}$ from b to c and Q_3 at $\dot{q}_3$ from a to d, between the same two conservative curves σ_1 and σ_2.

value σ_1, to another state c, on another Q-conservative curve specified by σ_2. Then since

$$\bar{d}Q = \phi(\dot{q}) f(\sigma) d\sigma,$$

it is seen that

$$\Delta Q = \phi(\dot{q}) \int_{\sigma_1}^{\sigma_2} f(\sigma) d\sigma \quad \text{at constant } \dot{q}. \qquad \text{(II-11)}$$

For any constant velocity process between two other points a to d, at a velocity $\dot{q}_3$ between the same two Q-conservative curves the energy transferred is

$$\Delta Q(\dot{q}_3) = \Delta Q_3 = \phi(\dot{q}_3) \int_{\sigma_1}^{\sigma_2} f(\sigma)\, d\sigma \quad \text{at constant } \dot{q}_3.$$

Taking the ratio of

$$\frac{\Delta Q}{\Delta Q_3} = \frac{\phi(\dot{q})}{\phi(\dot{q}_3)} = \frac{\text{a function of the velocity at which } \Delta Q \text{ is transferred}}{\text{same function of velocity at which } \Delta Q_3 \text{ is transferred}}.$$

Then the ratio of these two functions is defined by

$$\frac{\phi(\dot{q})}{\phi(\dot{q}_3)} = \frac{\Delta Q \text{(between } \sigma_1 \text{ and } \sigma_2 \text{ at } \dot{q}\text{)}}{\Delta Q_3 \text{(between } \sigma_1 \text{ and } \sigma_2 \text{ at } \dot{q}_3\text{)}}$$

or

$$\Delta Q = [\frac{\Delta Q_3}{\phi(\dot{q}_3)}]\ \phi(\dot{q}),$$

by choosing some appropriate velocity $\dot{q}_3$ then it follows that the energy transferred at constant velocity between two given Q-conservative curves decreases as $\phi(\dot{q})$ decreases, or the smaller the value of Q the lower the corresponding value of $\phi(\dot{q})$. When ΔQ is zero $\phi(\dot{q})$ is also zero. The corresponding velocity $\dot{q}_0$ such that $\phi(\dot{q}_0)$ is zero is the "absolute velocity". Therefore, if a system undergoes a constant velocity process between two Q-conservative curves without an exchange of energy, the velocity at which this takes place is called the absolute velocity.

d. The Concept of Entropy

In a system of two independent variables, all states accessible from a given initial state by reversible Q-conservative processes lie on a $\sigma(\dot{q},q)$ curve. The entire $(\dot{q},q)$ space may be conceived as being filled by many non-intersecting curves of this kind, each corresponding to a different value of σ. In a reversible non Q-conservative process involving a transfer of energy $\bar{d}Q$, a system in a state represented by a point lying on a surface σ will change until its state point lies on another surface $\sigma+d\sigma$. Then

$$\bar{d}Q = \lambda d\sigma,$$

where $1/\lambda$, the integrating factor of $\bar{d}Q$, is given by

$$\lambda = \phi(\dot{q})f(\sigma),$$

and therefore

$$\bar{d}Q = \phi(\dot{q})f(\sigma)d\sigma$$

or

$$\frac{\bar{d}Q}{\phi(\dot{q})} = f(\sigma)d\sigma .$$

Since σ is an actual function of $\dot{q}$ and q the right-hand member is an exact differential, which may be denoted by dS; and

$$dS = \frac{\bar{d}Q}{\phi(\dot{q})} , \tag{II-12}$$

where S is the mechanical entropy of the system and the process is a reversible one.

The dynamical second law may be used to prove equivalent of Clausius' theorem, which is stated here without proof.

Theorem: In any cyclic transformation throughout which the velocity is defined, the following inequality holds:

$$\oint \frac{\bar{d}Q}{\phi(\dot{q})} \leq 0 , \tag{II-13}$$

where the integral extends over one cycle of the transformation. The equality holds if the cyclic transformation is reversible. Then for an arbitrary transformation

$$\int_A^B \frac{\bar{d}Q}{\phi(\dot{q})} \leq S(B) - S(A) , \tag{II-14}$$

with the equality holding if the transformation is reversible. The proof of this statement may be seen by letting R and I denote respectively any reversible and any irreversible path joining A to B, as shown in Figure 7.

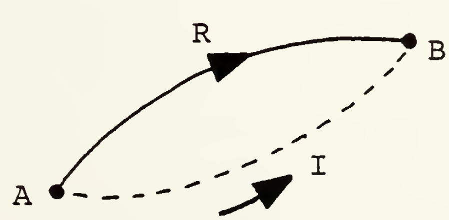

Figure 7

For path R the assertion holds by definition of S. Now consider the cyclic transformation made up of I plus the reverse of R. From Clausius' theorem

$$\int_I \frac{\bar{d}Q}{\phi} - \int_R \frac{\bar{d}Q}{\phi} \leq 0,$$

or

$$\int_I \frac{\bar{d}Q}{\phi} \leq \int_R \frac{\bar{d}Q}{\phi} \equiv S(B) - S(A). \qquad \text{(II-15)}$$

Another result of the dynamical second law is that the mechanical entropy of an isolated ($\bar{d}Q = 0$) system never decreases. This can be seen since an isolated system cannot exchange energy with the external world since $\bar{d}Q = 0$ for any transformation. Then by the previous property of the entropy

$$S(B) - S(A) \geq 0$$

where the equality holds if the transformation is reversible.

One consequence of the second law is that of all the possible transformations from one state A to another state B the one defined as the change in the entropy is the one for which the integral

$$I \equiv \int_A^B \frac{\bar{d}Q}{\phi} \qquad \text{(II-16)}$$

is a maximum. Thus

$$\mathcal{S}(B) - \mathcal{S}(A) \equiv \text{maximum } I = \max \int_A^B \left(\frac{1}{\phi} \frac{\bar{d}Q}{d\tau}\right) d\tau,$$

where τ is a parameter which indicates position along the path from A to B, or

$$\mathcal{S}(B) - \mathcal{S}(A) \equiv \max \int_A^B \left(\frac{1}{\phi} \frac{d\mathcal{U}}{d\tau} - \frac{\mathcal{F}}{\phi} \frac{dq}{d\tau}\right) d\tau;$$

If

$$\mathcal{U} = \mathcal{U}(\tau, q, \dot{q}, \frac{d\dot{q}}{d\tau})$$

where $\dot{q}^i = dq^i/dt$, then the change in the entropy is given by the integral

$$\Delta\mathcal{S} = \int_A^B \left(\frac{1}{\phi} \frac{d\mathcal{U}}{d\tau} - \frac{\mathcal{F}}{\phi} \frac{dq}{d\tau}\right) d\tau.$$

The $\dot{q}$ and q which maximize $\Delta\mathcal{S}$ will be denoted as $\dot{x}$ and x then, with

$$\mathcal{U} = \mathcal{U}(x, \dot{x})$$

$$\mathcal{F}_i = \mathcal{F}_i(x, \dot{x})$$

$$\phi = \phi(\dot{x})$$

the x and $\dot{x}$ are given by the solution of the system of equations

$$\frac{d}{d\tau}\left(\frac{\partial G}{\partial x'}\right) - \frac{\partial G}{\partial x} = 0$$

$$\frac{d}{d\tau}\left(\frac{\partial G}{\partial \dot{x}'}\right) - \frac{\partial G}{\partial \dot{x}} = 0 \qquad \text{(II-17)}$$

where

$$G = \left(\frac{1}{\phi}\right)\left[\frac{\partial U}{\partial \tau} - F_i \frac{dx}{d\tau}\right] \quad \text{and} \quad x' = \frac{dx}{d\tau} \quad \text{and} \quad \dot{x}' = \frac{d\dot{x}}{d\tau}.$$

Thus the dynamical second law provides an answer to the question that is not contained within the scope of the first law: In what direction does a process take place? The answer is that a process always takes place in such a direction as to cause an increase of the mechanical entropy in the universe. In the case of an isolated system, it is the entropy of the system that tends to increase. To find out, therefore, the equipoise state of an isolated one dimensional system, it is necessary merely to express the entropy as a function of q and $\dot{q}$ and to apply the usual rules of calculus to render the function a maximum. When the system is not isolated there are other entropy changes to be taken into account. It can be shown (Section IV-A) that there exists another energy function which refers to the system alone such that the equilibrium state of a non-isolated system is found by locating the minimum of this function.

D. THIRD LAW

The dynamical second law enables the mechanical entropy of a system to be defined up to an arbitrary additive constant. The definition depends on the existence of a reversible transformation connecting an arbitrarily chosen reference state 0 to the state under consideration. Such a reversible transformation always exists if both 0 and A lie on one sheet of the equation of the state surface. If two different systems are considered the equation of the state surface may consist of several disjoint sheets. In such cases the kind of reversible path previously mentioned may not exist. Therefore the second law does not uniquely determine the difference in entropy of two states A and B, if A defines a state of one system and B the state of another. For this determination a dynamical third law is needed. The dynamical third law may be stated, "The mechanical entropy of a system at the absolute velocity is a universal constant, which may be taken to be zero." In the case of a purely thermodynamic system the absolute quantity is the absolute zero temperature, while for a mechanical system the absolute quantity is the absolute velocity.

The dynamical third law implies that any energy capacity of a system must vanish at the absolute velocity. To see this, let R be any reversible path connecting a state of the system at the absolute velocity $\dot{q}_0$ to the state A, whose entropy is to be found. Let $C_R(\dot{q})$ be the energy capacity of the system along the path R. Then, by the second law,

$$S(A) = \int_{\dot{q}_0}^{\dot{q}_A} C_R(\dot{q}) \frac{d\dot{q}}{\phi(\dot{q})} .$$

But according to the third law,

$$\lim_{\dot{q}_A \to \dot{q}_0} S(A) \to 0.$$

Hence it follows that

$$\lim_{\dot{q} \to \dot{q}_0} C_R(\dot{q}) \to 0 \qquad \text{(II-18)}$$

In particular, C_R may be C_q or C_F.

III. GENERAL MAXWELL AND ENERGY RELATIONS

In thermodynamics a discussion of equilibrium and stability conditions is best done if the enthalpy, Helmholtz's, and Gibb's functions are defined first. Therefore the mechanical analoques of these functions are defined here.

Each branch of physics such as thermodynamics and particle dynamics has its own list of developed procedures. If both branches can be described by the same basic dynamic laws then the procedures developed in thermodynamics may prove to be useful in particle dynamics and vice-versa. Once the mechanical entalpy, mechanical Helmholtz's and mechanical Gibbs' functions are defined it is then easy to write down the resulting mechanical Maxwell and mechanical energy capacity relations. Therefore, while these relations are not used later in this investigation, they are presented here.

To begin the development of the Maxwell relations, the mechanical entropy was defined as

$$dS \equiv \frac{\bar{d}Q}{\phi(\dot{q})}, \qquad \text{(III-1)}$$

then, since $\bar{d}Q = dU - Fdq$,

$$dS = \frac{dU}{\phi} - \frac{F}{\phi}\, dq, \qquad \text{(III-2)}$$

where

$$dU = \phi(\dot{q})\ dS + F dq. \tag{III-3}$$

Define the mechanical enthalpy as

$$H \equiv U - Fq, \tag{III-4}$$

then

$$dH = \phi(\dot{q})\ dS - q dF, \tag{III-5}$$

therefore

$$\left(\frac{\partial H}{\partial S}\right)_F = \phi(\dot{q}) \qquad \text{and} \qquad \left(\frac{\partial H}{\partial F}\right)_S = -q. \tag{III-6}$$

The mechanical Helmholtz's function can be defined as

$$K \equiv U - \phi(\dot{q}) S, \tag{III-7}$$

and

$$dK = dU - \frac{d\phi(\dot{q})}{d\dot{q}} S d\dot{q} - \phi(\dot{q})\ dS,$$

or, with

$$\phi'(\dot{q}) = \frac{d\phi}{d\dot{q}}$$

$$dK = -S\phi'(\dot{q})\, d\dot{q} - F dq \, , \tag{III-8}$$

which leads to

$$\left(\frac{\partial K}{\partial \dot{q}}\right)_q = -S\phi'(\dot{q}) \quad \text{and} \quad \left(\frac{\partial K}{\partial q}\right)_{\dot{q}} = \phi(\dot{q})\, F. \tag{III-9}$$

The mechanical Gibb's function may be defined as

$$G \equiv H - \phi(\dot{q})\, S \, , \tag{III-10}$$

then

$$dG = -\phi'(\dot{q})S\, d\dot{q} + q\, dF, \tag{III-11}$$

so that

$$\left(\frac{\partial G}{\partial \dot{q}}\right)_F = -\phi'(\dot{q})\, S \quad \text{and} \quad \left(\frac{\partial G}{\partial F}\right)_{\dot{q}} = q. \tag{III-12}$$

From the differential equations (III-3), (III-5), (III-8) and (III-11) the Maxwell relations for a mechanical system may be written:

$$\begin{aligned}
\phi'(\dot{q})\left(\frac{\partial \dot{q}}{\partial q}\right)_S &= \left(\frac{\partial F}{\partial S}\right)_q \\
\phi'(\dot{q})\left(\frac{\partial \dot{q}}{\partial F}\right) &= -\left(\frac{\partial q}{\partial S}\right)_F \\
\phi'(\dot{q})\left(\frac{\partial S}{\partial q}\right)_{\dot{q}} &= -\left(\frac{\partial F}{\partial \dot{q}}\right)_q \\
\phi'(\dot{q})\left(\frac{\partial S}{\partial F}\right)_{\dot{q}} &= \left(\frac{\partial q}{\partial \dot{q}}\right)_F
\end{aligned} \tag{III-13}$$

The energy capacity at the position q can be defined as

$$C_q \equiv \left(\frac{\bar{d}Q}{d\dot{q}}\right)_q = \phi(\dot{q})\left(\frac{\partial S}{\partial \dot{q}}\right)_q . \tag{III-14}$$

Define the energy capacity with a constant force as

$$C_F \equiv \left(\frac{\bar{d}Q}{d\dot{q}}\right)_F = \Phi(\dot{q})\left(\frac{\partial S}{\partial \dot{q}}\right)_F, \tag{III-15}$$

then

$$(C_q - C_F) = \frac{\phi(\dot{q})}{\phi'(\dot{q})}\left(\frac{\partial q}{\partial \dot{q}}\right)_F\left(\frac{\partial F}{\partial \dot{q}}\right)_q , \tag{III-16}$$

and

$$\frac{C_F}{C_q} = \frac{\left(\frac{\partial F}{\partial q}\right)_S}{\left(\frac{\partial F}{\partial q}\right)_{\dot{q}}} . \tag{III-17}$$

IV. EQUIPOISE AND STABILITY

This section derives the equipoise and stability conditions for the mechanical system. These are the conditions required to satisfy the dynamical laws and lead to quadratic forms which provide natural metrics in the sense that adoption of these quadratic forms as the metric for a description of the system motion ensures that the resulting motion always satisfies the stability conditions.

The words and symbols used during the derivation of the equipoise and stability conditions are those used with a mechanical system. It is not difficult to see that the simple replacement of those words and symbols with their appropriate thermodynamic analogues yields the thermodynamic equilibrium and stability conditions.

A. EQUIPOISE CONDITIONS

To discuss dynamic equilibrium the criteria for an equipoise must be established. To establish the criteria for equipoise consider Clausius' theorem

$$\int_{AI}^{B} \frac{\bar{d}Q}{\phi} - \int_{AR}^{B} \frac{\bar{d}Q}{\phi} \leq 0,$$

or

$$\int_{AI}^{B} \frac{\bar{d}Q}{\phi} \leq \int_{AR}^{B} \frac{\bar{d}Q}{\phi} \equiv \mathcal{S}(B) - \mathcal{S}(A).$$

For a Q-conservative system $\bar{d}Q = 0$, then

$$\Delta S \geq 0,$$

or

$$S(B) \geq S(A).$$

Therefore the mechanical entropy tends toward a maximum so that spontaneous changes in a Q-conservative system will always be in the direction of increasing mechanical entropy. The application of this condition for a number of special cases will be considered next.

1. $\Delta Q = 0$ and Constant F.

The mechanical entropy change must be given by

$$\Delta S \geq \frac{\Delta Q}{\phi} = 0,$$

if the process is to be a spontaneous one. Now by the first law

$$\Delta Q = \Delta U - F\Delta q.$$

Therefore

$$\phi\Delta S \geq \Delta U - F\Delta q,$$

which is analogous to the Clausius inequality in thermodynamics.

Now consider a virtual displacement $(\mathcal{U},q) \to (\mathcal{U}-\delta\mathcal{U},q+\delta q)$, which implies a variation $\mathcal{S} \to \mathcal{S}+\delta\mathcal{S}$ away from equipoise. The restoration of equipoise from the varied state $(\mathcal{U}+\delta\mathcal{U},q+\delta q) \to (\mathcal{U},q)$ will then certainly be a spontaneous process, and by the Clausius inequality

$$\phi(-\delta\mathcal{S}) > -(\delta\mathcal{U} - \mathcal{F}\delta q).$$

Hence, for variations away from equipoise, the general inequality

$$\delta\mathcal{U} - \mathcal{F}\delta q - \phi\delta\mathcal{S} > 0, \qquad \text{(IV-1)}$$

must hold. The inequality sign is reversed from the sign in Clausius' inequality because hypothetical variations δ away from equipoise are considered rather than real changes Δ toward equipoise.

Now consider the special case where $\delta\mathcal{U} = 0$ and $\delta q = 0$, then

$$(\delta\mathcal{S})_{\mathcal{U},q} < 0. \qquad \text{(IV-2)}$$

Therefore, at equipoise the entropy is a maximum with respect to all variations which leave the position and energy of the system constant, which implies that all variations must be within the system.

If $\delta S = 0$ and $\delta q = 0$ then

$$(\delta U)_{S,q} > 0 \,, \qquad \text{(IV-3)}$$

or at equipoise the energy of the system is a minimum with respect to variations at constant entropy.

Formally the criterion given by equation (IV-3) follows from equation (IV-1) just as readily as the condition (IV-2) does. To prove this equivalence suppose equation (IV-2) were true and equation (IV-3) were not. The violation of (IV-3) is a variation α such that

$$\delta U_\alpha < 0 \qquad \text{when} \qquad \delta S_\alpha = 0.$$

Now a subsequent variation β can always be found whereby both U and S increase, simply by letting some of the absorbed work dissipate within the system. Thus

$$\delta U_\beta > 0; \qquad \delta S_\beta > 0.$$

The latter step could be arranged so that the total variations would be

$$\delta U_{\alpha+\beta} = 0; \qquad \delta S_{\alpha+\beta} > 0,$$

which contradicts equation (IV-2).

While inequality (IV-2) identifies the equipoise of a Q-conservative system as a maximum of entropy, inequality (IV-3) shows that equipoise is a state of minimum system energy.

2. $\Delta Q = 0$ and Variable F.

Suppose now that the force is not held constant but ΔQ is still zero. The entropy will still be a maximum at equipoise, however, there is now a different subsidiary condition. Not the energy of the system U but U plus a certain mechanical potential energy representing the coupling to the surroundings, is to be constant under the variations. If the coupling is achieved by the force only then this mechanical potential energy is just the negative of Fq and hence the mechanical enthalpy

$$H = U - Fq$$

must be kept constant under virtual displacements. Therefore, corresponding to equation (IV-2) and (IV-3) are the conditions

$$(\delta S)_{H,F} < 0, \qquad \text{(IV-4)}$$

and

$$(\delta H)_{S,F} > 0. \qquad \text{(IV-5)}$$

To prove this formally replace U by $H + Fq$ and use the difference relations

$$\delta U = \delta H + \delta(Fq)$$

$$\delta(Fq) = (F + \delta F)(q + \delta q) - Fq \qquad \text{(IV-6)}$$

$$= F\delta q + q\delta F + \delta F\delta q.$$

Inserting the above into equation (IV-1) results in

$$\delta H + F\delta q + q\delta F + \delta F\delta q - F\delta q - \phi\delta S > 0$$

or

$$\phi\delta S - \delta H - q\delta F - \delta F\delta q < 0, \qquad \text{(IV-7)}$$

from which inequalities (IV-4) and (IV-5) follow.[1] Thus at constant force the mechanical entropy is maximum at constant mechanical enthalpy and the mechanical enthalpy is minimum at constant mechanical entropy. For systems at constant force the mechanical enthalpy H plays a role analogous to that of the system energy U for systems at constant position.

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[1]The reason for retaining the term $\delta F\delta q$ is that, although it does not affect the equipoise conditions (IV-4) and (IV-5), the variations in Clausius' inequality are not necessarily infinitesimal. The stability problem is one instance in which this must be remembered (see next section).

3. Variable Q, ϕ = constant

Now consider a non Q-conservative system ($\bar{d}Q \neq 0$). But assume that ϕ remains constant. Then for $\delta q = 0$ equation (IV-1) implies that the mechanical Helmholtz' free energy,

$$K \equiv U - \phi S, \qquad \text{(IV-8)}$$

is a minimum, since K is then positive for a variation from equipoise. Similarly, for equipoise at constant force, equation (IV-7) implies that the mechanical Gibb's free energy,

$$G \equiv H - \phi S = U - \phi S - qF \qquad \text{(IV-9)}$$

is a minimum. The equipoise conditions may then be written

$$(\delta K)_{\phi,q} > 0 \qquad \text{and} \qquad (\delta G)_{\phi,F} > 0, \qquad \text{(IV-10)}$$

respectively. K may also be called mechanical "free energy at the position q", and G the mechanical "free energy at constant generalized force."

4. General Equipoise Conditions

It was shown in the previous section that the enthalpy or free energy are a minimum at equipoise. Each condition is a special case of the general inequality (IV-1). To obtain a general condition for equipoise consider the inequality a little further. In a spontaneous process,

$$\phi\Delta S \geq \Delta Q_{Rev} = \Delta U + \text{work done by the system.} \tag{IV-11}$$

The "work" consists of two parts. One part is the work done by the negative of the force F. It may be positive or negative but it is inevitable. Only the rest is free energy, which is available for some useful work. This latter part may be written as

$$A = \Delta Q_{Rev} - \Delta U + F\Delta q. \tag{IV-12}$$

The maximum of A according to (IV-11) is

$$A_{max} = \phi\Delta S - \Delta U + F\Delta q, \tag{IV-13}$$

which is obtained when the process is conducted reversibly.

The least work, δA_{min}, required for a displacement from equipoise must be exactly equal to the maximum work in the converse process whereby the system proceeds spontaneously from the "displaced" state to equipoise (otherwise a perpetual motion machine may be constructed). Corresponding to equation (IV-13) then, is

$$\delta A_{min} = \delta U - F\delta q - \phi\delta S. \tag{IV-14}$$

All equipoise criteria can therefore be condensed into one:

$$\delta A_{min} \geq 0. \tag{IV-15}$$

In words: At equipoise the mechanical free energy is a minimum. Any displacement from this state requires work.

Table 1 is a tabulation of the equipoise conditions for the various special cases and indicates the applicability of the general equipoise conditions to each special case.

B. STABILITY

First order conditions such as $\delta \mathcal{S} = 0$, $\delta \mathcal{K} = 0$, and so on are necessary but not sufficient for equipoise. To decide whether or not an equipoise is stable, the inequality sign in (IV-1) must be ensured.

1. Stability with q and $\mathcal{S}$ as Independent Variables.

Consider the terms of second order in small displacements beginning with the general condition

$$\delta \mathcal{U} - \mathcal{F}\delta q - \phi\delta \mathcal{S} > 0. \qquad \text{(IV-16)}$$

Choose $\mathcal{U} = \mathcal{U}(q,\mathcal{S})$, which, because of the identity

$$d\mathcal{S} = \frac{d\mathcal{U}}{\phi} - \frac{\mathcal{F}}{\phi}\,dq,$$

or

$$\phi d\mathcal{S} = d\mathcal{U} - \mathcal{F}dq,$$

TABLE OF SPECIAL CASES

SPECIAL CASES:

1. $\Delta Q = 0$; F = constant

   a. $\delta U = 0$; $\delta q = 0$ implies $(\delta S)_{U,q} < 0$ max.

      S at constant U and q.

   b. $\delta S = 0$; $\delta q = 0$ implies $(\delta U)_{S,q} > 0$ min.

      S at constant S and q.

2. $\Delta Q = 0$; $H \equiv U - Fq$

   a. $\delta H = 0$; $\delta F = 0$ implies $(\delta S)_{H,F} < 0$ max.

      S at constant H and F.

   b. $\delta S = 0$; $\delta F = 0$ implies $(\delta H)_{S,F} > 0$ min.

      H at constant S and F.

3. $\Delta Q = 0$; $K \equiv U - \phi S$; $G \equiv H - \phi S$;

   a. $\delta q = 0$; implies $(\delta K)_{q,\phi} > 0$ min. K at constant q.

   b. $\delta F = 0$; implies $(\delta G)_{F,\phi} > 0$ min. G at constant F.

GENERAL EQUIPOISE CONDITION $(\delta A) > 0$

$\delta A \equiv \delta U - F\delta q - \phi\delta S$

1a. $\delta A = -\phi\delta S > 0$ implies $(\delta S)_{U,q} < 0$

1b. $\delta A = \delta U > 0$ implies $(\delta U)_{S,q} > 0$

2a. $\delta A = \delta H + q\delta F - \phi\delta S = -\phi\delta S > 0$ implies $(\delta S)_{H,F} < 0$

2b. $\delta A = \delta H > 0$ implies $(\delta H)_{S,F} > 0$

3a. $\delta A = \delta K + S\delta\phi - F\delta q = \delta K > 0$ implies $(\delta K)_{q,\phi} > 0$

3b. $\delta A = \delta G + q\delta F + S\delta\phi = \delta G > 0$ implies $(\delta G)_{\phi,F} > 0$

Table 1. A tabulation of the equipoise conditions for various special cases.

is a natural choice of the independent variables, and expand δU in powers of δq and δS

$$\delta U = \phi\delta S + F\delta q + \frac{1}{2}\left(\frac{\partial^2 U}{\partial q^2}\delta q^2 + 2\frac{\partial^2 U}{\partial q \partial S}\delta q \delta S + \frac{\partial^2 U}{\partial S^2}\delta S^2\right) + \text{terms of third order} + \ldots \qquad \text{(IV-17)}$$

The inequality (IV-1) then shows that in (IV-17)

Second order terms + third order terms + ... > 0. (IV-18)

Retaining only the second order terms, the criterion of stability is that a quadratic differential form be positive definite;

$$\frac{\partial^2 U}{\partial q^2}\,\delta q^2 + 2\,\frac{\partial^2 U}{\partial q \partial S}\,\delta q \delta S + \frac{\partial^2 U}{\partial S^2}\,\delta S^2 > 0. \qquad \text{(IV-19)}$$

If this is to hold true for arbitrary variations in δq and δS, the coefficients must satisfy the following:

$$\frac{\partial^2 U}{\partial q^2} > 0; \quad \frac{\partial^2 U}{\partial S^2} > 0; \quad \frac{\partial^2 U}{\partial S^2}\,\frac{\partial^2 U}{\partial q^2} - \left(\frac{\partial^2 U}{\partial q \partial S}\right)^2 > 0. \qquad \text{(IV-20)}$$

2. Stability with q and $\dot{q}$ as Independent Variables

A quadratic form in δq and $\delta\dot{q}$ may be found by considering

$$K = U - \phi S$$

so that

$$\delta K = \delta U - \phi\delta S - \frac{d\phi}{d\dot{q}} S\delta\dot{q} - \frac{d\phi}{d\dot{q}} \delta S\delta\dot{q}.$$

The terms $\delta S\delta\dot{q}$ cannot be neglected because in Clausius' inequality, which is the actual stability condition, the variations are finite, therefore, from equation (IV-16) the following is obtained:

$$\delta K + \phi\delta S + \frac{d\phi}{d\dot{q}}(S + \delta S)\delta\dot{q} - F\delta q - \phi\delta S > 0,$$

$$\delta K + \frac{d\phi}{d\dot{q}}S\delta\dot{q} + \frac{d\phi}{d\dot{q}}\delta S\delta\dot{q} - F\delta q > 0.$$

Expanding in powers of $\delta\dot{q}$ and δq

$$\delta K = F\delta q - \frac{d\phi}{d\dot{q}}S\delta\dot{q} + \frac{1}{2}\frac{\partial^2 K}{\partial q^2}\delta q^2 + \frac{\partial^2 K}{\partial q\partial\dot{q}}\delta q\delta\dot{q} + \frac{1}{2}\frac{\partial^2 K}{\partial\dot{q}^2}\delta\dot{q}^2 + \ldots$$

$$\delta S\delta\dot{q} = \frac{1}{\phi}\frac{\partial U}{\partial\dot{q}}\delta\dot{q}^2 + \frac{1}{\phi}\left(\frac{\partial U}{\partial q} - F\right)\delta q\delta\dot{q}$$

$$\frac{\partial K}{\partial\dot{q}} = \frac{\partial U}{\partial\dot{q}} - \frac{\partial\phi}{\partial\dot{q}}S - \frac{\phi}{\phi}\frac{\partial U}{\partial\dot{q}} = -\frac{d\phi}{d\dot{q}}S\ ;$$

$$\frac{\partial K}{\partial q} = F.$$

Therefore

$$\frac{\partial^2 K}{\partial\dot{q}\partial q} = \frac{\partial F}{\partial\dot{q}} = -\frac{\partial\phi}{\partial\dot{q}}\left(\frac{1}{\phi}\right)\left(\frac{\partial U}{\partial q} - F\right),$$

and

$$\frac{\partial^2 K}{\partial \dot{q}^2} = - \frac{\partial^2 \phi}{\partial \dot{q}^2} S - \frac{d\phi}{d\dot{q}} (\frac{1}{\phi}) \frac{\partial U}{\partial \dot{q}};$$

then

$$(\frac{d\phi}{d\dot{q}}) \delta S \delta \dot{q} = -(\frac{\partial^2 \phi}{\partial \dot{q}^2} S + \frac{\partial^2 K}{\partial \dot{q}^2}) (\delta \dot{q})^2 - \frac{\partial^2 K}{\partial \dot{q} \partial q} \delta q \delta \dot{q},$$

and the quadratic form in $\delta\dot{q}$ and δq is

$$\frac{1}{2} \frac{\partial^2 K}{\partial q^2} (\delta q)^2 + \frac{\partial^2 K}{\partial q \partial \dot{q}} \delta q \delta \dot{q} + \frac{1}{2} \frac{\partial^2 K}{\partial \dot{q}^2} (\delta \dot{q})^2 - \frac{\partial^2 K}{\partial \dot{q}^2} (\delta \dot{q})^2 - \frac{d^2 \phi}{d\dot{q}^2} S (\delta \dot{q})^2 - \frac{\partial^2 K}{\partial \dot{q} \partial q} \delta q \delta \dot{q} > 0,$$

or

$$\frac{\partial^2 K}{\partial q^2} (\delta q)^2 - (\frac{\partial^2 K}{\partial \dot{q}^2} + 2 \frac{d^2 \phi}{\dot{q}^2} S) (\delta \dot{q})^2 > 0. \qquad \text{(IV-21)}$$

Since $(\frac{\partial K}{\partial q})_{\dot{q}} = F$ then

$$\frac{\partial^2 K}{\partial q^2} = (\frac{\partial F}{\partial q})_{\dot{q}} > 0. \qquad \text{(IV-22)}$$

Another quadratic form may be obtained when the independent system variables are taken to be other than (q, S) or $(\dot{q}, q)$. The quadratic form given as equation (IV-19) demonstrates that the "natural" variables for the system energy are space coordinates and mechanical entropy. The quadratic

form, equation (IV-21), shows that space and velocity are the "natural" variables for the mechanical Helmholtz function.

V. ISENTROPIC MECHANICS

The objective of the investigation is to determine whether or not the logic structure of classical thermodynamics could yield dynamical laws which would produce equations of motion containing existing dynamical theories and in addition provide a directivity. The logical procedure to obtain this objective, given the development up to the end of Section IV, is to adopt as the metric of the system one of the quadratic forms which will ensure the system's stability. Suppose that the quadratic form, equation (IV-19), were to be adopted as the metric. This would say that the system is described in a space-mechanical entropy manifold. The idea of describing particle dynamics in such a manifold is not known to have been previously investigated. The suggestion that particle dynamics be described in a manifold other than the space-time manifold of relativistic dynamics immediately raises a number of questions.

These questions prompt a deviation from the logical procedure. By taking the time here to consider some of the familiar procedures of classical and relativistic dynamics, consistency between them and the dynamics proposed here can be demonstrated. The role of the integrating factor, absolute velocity and the mechanical entropy can also be seen.

The dynamics of both Newtonian and relativistic mechanics is time symmetrical. This suggests that if the dynamics

provided by the dynamic laws presented here are to be consistent with these theories then the systems for which this consistency may exist are reversible isentropic systems. Therefore only isentropic are considered in this section.

A. CLASSICAL MECHANICS

Classical mechanics describes the motion of a system, which could be a particle, for which the energy of the system is a constant. The equations of motion may be obtained using Hamilton's principle. These equations of motion yield trajectories resulting from the action of forces; they may also be obtained from the principle of least action. When the action integral is treated as a variational problem with variable end points the method of Lagrangian multipliers yields the same equations as does Hamilton's principle. However, if the variational problem is transformed to a new space in which the new variational problem has fixed end points, then the metric for this space is displayed, and the equations of motion are geodesics in this space.

In classical mechanics the principle of least action as formulated by Lagrange[3] has the integral form

$$A = \int_{P_1}^{P_2} m\bar{v} \cdot d\bar{s} \ . \tag{V-1}$$

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[3]Sokolnikoff, I.S., Tensor Analysis Theory and Applications to Geometry and Mechanics of Continua, pp. 230-232, 1964.

In curvilinear coordinates the integral assumes the form

$$A = \int_{P_1}^{P_2} mg_{\alpha\beta} \frac{dx^\alpha}{dt} dx^\beta = \int_{t(P_1)}^{t(P_2)} mg_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} dt$$

or defining

$$T \equiv \frac{m}{2} g_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt}$$

the integral becomes

$$A = \int_{t(P_1)}^{t(P_2)} 2T dt.$$

Then the principle of least action may be stated as:

Of all curves C' passing through P_1 and P in the neighborhood of the trajectory C, which are traversed at a rate such that, for each C', for every value of t, T + V = F, that one for which the action integral A is stationary is the trajectory of the particle.

In Appendix B the transformation of variables is carried out so that the metric is displayed. The result of this transformation is the metric

$$dS^2 = h_{\alpha\beta}\, dx^\alpha\, dx^\beta \qquad \text{(B-6)}$$

where

$$h_{\alpha\beta} = 2m(E_0 - V) g_{\alpha\beta}.$$

Suppose that this classical system is associated with the concepts presented in Section II. The energy of the system in classical mechanics is a constant of the motion and therefore the change in kinetic energy is the negative of the change in potential energy, which may be written as

$$dH = dT + dV = 0.$$

However for conservative forces dH is a perfect differential. Therefore, for a one-dimensional system the force is a function of position only.

This suggests the association of the classical energy of the system, H, with the system energy, $\mathcal{U}$, which is also a perfect differential. Now if the system is isentropic then this association leads to the relation

$$d\mathcal{S} = 0 = \frac{\bar{d}Q}{\phi} = \frac{d\mathcal{U}}{\phi} - \frac{\mathcal{F}}{\phi}\,dq.$$

But if $d\mathcal{U} = dH = 0$, then $\mathcal{F}$ must be zero.

Considering the quadratic form, equation (IV-19), for an isentropic system it can be seen that the only term left is a space term which is consistent with the space metric of classical mechanics given by equation (B-6) of Appendix B. Thus an isentropic system, for which the $\mathcal{F}$ is negligible, is consistent with a classical conservative system. The

mechanical entropy does not become involved for such a system because if $\mathcal{F}$ is negligible then $\bar{d}Q = d\mathcal{U}$. Therefore $\bar{d}Q$ must be a perfect differential. If $\bar{d}Q$ is a perfect differential there is no need to look for an integrating factor.

B. RELATIVISTIC MECHANICS

In the special theory of relativity Einstein sought to put Newtonian mechanics into a form which would leave the speed of light invariant. The resulting dynamics exhibits the notion of a unique velocity in a similar sense to the previously defined absolute velocity. The modification required the motion to be such that

$$\int_{t_1}^{t_2} \left(\frac{m\dot{q}\delta\dot{q}}{\sqrt{1-\dot{q}^2/c^2}} + F\delta q\right) dt = 0,$$

where F is a force which is a function of position only.

The factor $1 - \dot{q}^2/c^2$ displays the qualities required of the integrating factor $\phi(\dot{q})$. Therefore consider a modification of Hamilton's principle in terms of the system energy $\mathcal{U}$, the force $\mathcal{F}$ and the integrating factor ϕ. The modified statement then would be that the motion be such that

$$\int_{t_1}^{t_2} \left(\frac{\delta\mathcal{U}}{\phi} + \frac{\mathcal{F}}{\phi}\, q\right) dt = 0.$$

It can be seen that if $\phi(\dot{q}) = 1$ then $\mathcal{F}$ must be a function only of q and classical mechanics results. It will be shown that if $\phi(\dot{q}) = \sqrt{1 - \dot{q}^2/c^2}$ relativistic mechanics is obtained.

Now for an isentropic system

$$dS = \frac{\bar{d}Q}{\phi} = 0 = \frac{dU}{\phi} - \frac{F}{\phi}\, dq,$$

or

$$\int_A^B \frac{dU}{\phi} = \int_A^B \frac{F}{\phi}\, dq.$$

This would be the classical work-energy theorem if $\phi = 1$. For any ϕ

$$\frac{dU}{dt} = F\dot{q}.$$

If the system energy U is taken to be the kinetic energy and defined as

$$U \equiv \frac{1}{2} m\dot{q}^2 , \qquad \text{(V-1)}$$

then

$$m\ddot{q} = F,$$

or Newton's second law.

This tends to indicate that a modification of Hamilton's principle would apply to a system for which $dS = 0$. This modification would be to assume that for an isentropic system

the motion is given by the principle:

If a particle is at the point P_1 at the time t_1 and at the point P_2 at the time t_2 then the motion of the particle takes place in such a way that

$$\int_{t_1}^{t_2} \left(\frac{\delta\mathcal{U}}{\phi(\dot{q})} + \frac{F}{\phi(\dot{q})}\,\delta q\right) dt \equiv \delta \int_{t_1}^{t_2} \mathcal{L}\, dt = 0,$$

where $q = q(t)$ is the generalized coordinate of the particle along the trajectory and $q + \delta q$ is the coordinate along a varied path beginning at P_1 at the time t_1 and ending at P_2 at time t_2.

The hypothesis of the fundamental lemma of calculus of variations is that $\mathcal{L}$ be a real continuous function, therefore, the mixed second partial derivatives of $\mathcal{L}$ must be equal, or

$$\frac{\partial^2 \mathcal{L}}{\partial q \partial \dot{q}} = \frac{\partial^2 \mathcal{L}}{\partial \dot{q} \partial q}\ .$$

Now

$$d\mathcal{L} = \frac{1}{\phi(\dot{q})}\frac{\partial \mathcal{U}}{\partial \dot{q}}\, d\dot{q} + \frac{1}{\phi(\dot{q})}\left[\frac{\partial \mathcal{U}}{\partial q} + F\right] dq,$$

so that

$$\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{1}{\phi}\frac{\partial \mathcal{U}}{\partial \dot{q}} \qquad \text{and} \qquad \frac{\partial \mathcal{L}}{\partial q} = \frac{1}{\phi}\left[\frac{\partial \mathcal{U}}{\partial q} + F\right].$$

Then

$$\frac{\partial^2 L}{\partial q \partial \dot{q}} = \frac{1}{\phi} \frac{\partial^2 U}{\partial q \partial \dot{q}} = \frac{1}{\phi} \left(\frac{\partial^2 U}{\partial \dot{q} \partial q} + \frac{\partial F}{\partial \dot{q}}\right) - \frac{\phi'}{\phi^2}\left[\frac{\partial U}{\partial q} + F\right] .$$

This requires that

$$\frac{\partial F}{\partial \dot{q}} = \frac{\phi'}{\phi} \left(\frac{\partial U}{\partial q} + F\right). \tag{V-2}$$

However, dS is a perfect differential so that

$$\frac{\partial^2 S}{\partial q \partial \dot{q}} = \frac{\partial^2 S}{\partial \dot{q} \partial q} .$$

Since

$$dS = \frac{1}{\phi} \frac{\partial U}{\partial \dot{q}} d\dot{q} + \frac{1}{\phi} \left(\frac{\partial U}{\partial q} - F\right) dq$$

$$\frac{1}{\phi} \frac{\partial^2 U}{\partial q \partial \dot{q}} = \frac{1}{\phi} \left(\frac{\partial^2 U}{\partial \dot{q} \partial q} - \frac{\partial F}{\partial \dot{q}}\right) - \frac{\phi'}{\phi} \left(\frac{\partial U}{\partial q} - F\right),$$

or

$$\frac{\partial F}{\partial \dot{q}} = - \frac{\phi'}{\phi} \left(\frac{\partial U}{\partial q} - F\right). \tag{V-3}$$

In order that dS and dL both be perfect differentials at the same time then

$$\frac{\partial F}{\partial \dot{q}} = - \frac{\phi'}{\phi} \left(\frac{\partial U}{\partial q} - F\right) = \frac{\phi'}{\phi} \left(\frac{\partial U}{\partial q} + F\right).$$

Therefore

$$\frac{\partial U}{\partial q} = 0$$

and

$$\frac{\partial F}{\partial \dot{q}} = \frac{\phi'}{\phi} F$$

or

$$\frac{1}{F} \frac{\partial F}{\partial \dot{q}} = \frac{\phi'}{\phi}$$

which is a function of $\dot{q}$ only. Then

$$\frac{1}{\phi} \frac{\partial F}{\partial \dot{q}} - \frac{\phi'}{\phi^2} F = 0,$$

or

$$\frac{\partial}{\partial \dot{q}} \left(\frac{F}{\phi}\right) = 0,$$

which implies that

$$\frac{F}{\phi} = F(q)$$

which is a function of q only, and

$$\mathcal{F} = \phi(\dot{q})\ F(q) \tag{V-4}$$

for an isentropic system. Thus the factor ϕ cancels out of the differential expression for change in entropy so that effectively the force is not velocity dependent.

Suppose that the momentum is defined as

$$p \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{1}{\phi}\frac{\partial \mathcal{U}}{\partial \dot{q}}, \tag{V-5}$$

and the mass is defined as

$$m \equiv \frac{\partial^2 \mathcal{U}}{\partial \dot{q}^2}. \tag{V-6}$$

Then

$$d\mathcal{S} = p\ d\dot{q} - F(q)\ dq,$$

and

$$d\mathcal{L} = p\ d\dot{q} + F(q)\ dq.$$

The equations or motion would be

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] - \frac{\partial \mathcal{L}}{\partial q} = 0, \tag{V-7}$$

or

$$\frac{dp}{dt} = \frac{\partial L}{\partial q} = F(q).$$

If m = constant then $p = m\dot{q}/\phi$ and

$$dL = \frac{m\dot{q}}{\phi}\, d\dot{q} + F(q)\, dq$$

while

$$dS = \frac{m\dot{q}}{\phi}\, d\dot{q} - F(q)\, dq.$$

Then for $dS = 0$

$$S = \tau(\dot{q}) + V(q); \qquad L = \tau(\dot{q}) - V(q)$$

where

$$F(q) = -\frac{\partial V(q)}{\partial q}.$$

How then may $\phi(\dot{q})$ be determined? The precedence set by thermodynamics is to determine $\phi(\dot{q})$ experimentally. Experiments with a charged particle in a magnetic field, such as a mass spectrometer, show that

$$\phi(\dot{q}) = \sqrt{1 - \dot{q}^2/c^2} \quad . \tag{V-8}$$

This function satisfies the requirements for the integrating factor with c as the absolute velocity.

If this integrating factor is substituted into the equations of motion the resulting equations are

$$\frac{d}{dt}\left[\frac{m\dot{q}}{\sqrt{1-\dot{q}^2/c^2}}\right] = F(q),$$

then

$$dL = \frac{m\dot{q}\,d\dot{q}}{\sqrt{1-\dot{q}^2/c^2}} + F(q)\,dq,$$

$$L(\dot{q},q) - L(\dot{q}_0,q_0) = \int_{\dot{q}_0}^{\dot{q}} \frac{m\dot{q}\,d\dot{q}}{\sqrt{1-\dot{q}^2/c^2}} + \int_{q_0}^{q} F(q)\,dq$$

$$= mc^2\sqrt{1-\dot{q}^2/c^2}\,\Big|_{\dot{q}_0}^{\dot{q}} - V(q) + V(q_0).$$

If $L(\dot{q}_0,q_0) = L(0,\infty) = 0$, then

$$L(\dot{q},q) = mc^2\left[1-\sqrt{1-\dot{q}^2/c^2}\;\right] - V(q)\;. \qquad \text{(V-9)}$$

With the exception of the additive term mc^2 this is the form of the relativistic Lagrangian when m is interpreted as the rest mass, and since additive constants in the Lagrangian do not affect the equations of motion, this Lagrangian yields equations of motion consistent with the special theory of relativity.

The first integral of the equations of motion may be written as

$$\frac{d}{dt}\left[L - \dot{q}\frac{\partial L}{\partial \dot{q}}\right] = \frac{d}{dt}\left[L - \dot{q}p\right] = 0,$$

therefore

$$L - \dot{q}p = \text{constant}.$$

Then define this constant of the motion, which may be called a "Hamiltonian", by

$$H \equiv \dot{q}p - L. \tag{V-10}$$

Since the Lagrangian is given by

$$L = \int p d\dot{q} - V(q),$$

the Hamiltonian becomes

$$H = \dot{q}p - \int p d\dot{q} + V(q) = \int \dot{q} dp + V(q).$$

Then the Hamiltonian equations of motion may be written as

$$\frac{\partial H}{\partial q} = - F(q) = - \frac{dp}{dt}; \qquad \frac{\partial H}{\partial p} = \dot{q}. \tag{V-11}$$

For the particular Lagrangian

$$L = mc^2 \, [1 - \sqrt{1 - \dot{q}^2/c^2} \;] - V(q),$$

the Hamiltonian is

$$H = \frac{m\dot{q}}{\sqrt{1 - \dot{q}^2/c^2}} - mc^2[1 - \sqrt{1 - \dot{q}^2/c^2} \;] + V(q)$$

$$= mc^2 \left(\frac{1}{\sqrt{1 - \dot{q}^2/c^2}} - 1 \right) + V(q)$$

or

$$H = mc^2 \, (\gamma - 1) + V(q) \tag{V-12}$$

where

$$\gamma \equiv \frac{1}{\sqrt{1 - \dot{q}^2/c^2}} \; .$$

Then defining $E(\dot{q}) - E_0 \equiv mc^2(\gamma - 1)$ implies that

$$E^2 = E_0{}^2 + (pc)^2. \tag{V-13}$$

In the special theory of relativity the Hamiltonian, which is interpreted as the energy of the system when m is the rest mass and c is the speed of light, has the same form

as equation (V-12). In this dynamic theory, however, the Hamiltonian is not the energy of the system. The system energy is U and is given here by $U = \frac{1}{2}m\dot{q}^2$ since $\frac{\partial U}{\partial q} = 0$. H is a constant of the motion and therefore is at most only a constant different from the entropy of the system since the entropy is also a constant in this case.

These relativistic equations are symmetrical in time. They are the equations for a system with constant entropy and the time symmetry is consistent with reversibility.

Thus the concepts presented in Section II and a modified Hamilton's principle may be seen to produce dynamics consistent with special relativity. The roles of the absolute velocity, integrating factor, and mechanical entropy are also displayed.

C. GEOMETRIZATION

Transforming the integral of the classical least action principle to a space in which the variational problem had an integral with fixed end points displayed the metric. For an isentropic system this metric was seen to be consistent with the quadratic form required for stability. Now consider the metric of the space governed by the modification of Hamilton's principle in the previous section. It too is consistent with classical dynamics.

Impose the same requirements as in Section V-B, namely that dS and dL both be perfect differentials; so that $\frac{\partial U}{\partial q} = 0$. Then U is a function of velocity and is the kinetic energy

$$U = \frac{m}{2} g_{\alpha\beta} \frac{dq^\alpha}{dt} \frac{dq^\beta}{dt} \equiv \frac{1}{2}mv^2,$$

where the summation convention of tensor analysis is used. The $g_{\alpha\beta}$ are line element coefficients in the chosen coordinate system. The indices α and β take on the values 1, 2, or 3 to correspond to the spatial dimensions.

Since it is desired to expand the dimensionality of the system at this point it is necessary to discuss the extension of the argument of Caratheodory's to a higher dimension. For instance if for each dimension, q_α, requires a separate integrating factor $\phi_\alpha = \phi(q_\alpha)$; so that

$$dS_\alpha = \frac{\bar{d}Q_\alpha}{\phi_\alpha},$$

can the differential of the total mechanical entropy be written as

$$dS = \sum_\alpha dS_\alpha = \sum_\alpha \frac{\bar{d}Q_\alpha}{\phi_\alpha} = \frac{\bar{d}Q}{\phi} = \frac{1}{\phi} \sum_\alpha \bar{d}Q_\alpha ?$$

The proof that this can always be done was developed by Caratheodory and is presented in Appendix C. Therefore the mechanical entropy may be written as

$$dS = \frac{mg_{\alpha\beta}}{\phi} \frac{dq^\alpha}{dt} dq^\beta - \frac{F_\alpha}{\phi} dq^\alpha,$$

and in order for dS to be a perfect differential the ratio

F_α/ϕ must be a function of position only (see the discussion in Section V-A). The forces F_α must have the structure

$$F_\alpha = \phi \; f_\alpha(q). \tag{V-14}$$

Then consider the integral

$$I = \int_{P_1}^{P_2} \frac{\partial S}{\partial \dot{q}^\alpha} \, d\dot{q}^\alpha = \int_{t(P_1)}^{t(P_2)} \left(\frac{m}{\phi} \, g_{\alpha\beta} \frac{dq^\alpha}{dt} \frac{dq^\beta}{dt}\right) dt, \tag{V-15}$$

and the variational problem of minimizing this integral subject to the constant entropy requirement

$$S(\dot{q}^1,\dot{q}^2,\dot{q}^3,q^1,q^2,q^3) - S_0 = 0. \tag{V-16}$$

Again the variable limits of integration can be avoided by a change of variables and since, as noted in Section V-B the entropy can differ from the relativistic Hamiltonian by at most a constant, and recalling equation (V-12), for this situation

$$S = \tau(v^2) + V(q^1,q^2,q^3) \tag{V-17}$$

where ϕ is assumed to have the form $\phi = \phi(v^2)$. Then in principle the first term in equation (V-17) can be solved for

the velocity as a function of τ[4] or

$$\left(\frac{ds}{dt}\right)^2 = g_{\alpha\beta}\frac{dq^\alpha}{dt}\frac{dq^\beta}{dt} = [f(\tau)]^2; \quad (\alpha,\beta = 1,2,3) \tag{V-18}$$

The ratio $\bar{m}/\phi$ may then be expressed as a function of τ also

$$\bar{m}/\phi = G(\tau). \tag{V-19}$$

Since the entropy is a constant, equation (V-16) may be solved for τ so that

$$\tau = S_0 - \nu \tag{V-20}$$

Then

$$\bar{m}/\phi = G(S_0 - \nu) \tag{V-21}$$

and

$$g_{\alpha\beta}\frac{dq^\alpha}{dt}\frac{dq^\beta}{dt} = [f(S_0 - \nu)]^2. \tag{V-22}$$

Substituting equations (V-21) and (V-22) into the integral (V-15) gives a new integral

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[4]This is consistent with the defining equation (V-6) and the assumption of that section which was that $\frac{\partial^2 u}{\partial \dot{q}\alpha^2} \equiv m \equiv \text{constant}.$

$$I = \int_{t(p_1)}^{t(p_2)} G[S_0 - \ ][f(S_0 - \nu]^2 \, dt, \qquad \text{(V-23)}$$

with the integrand independent of the time. The varied paths can be parameterized so that C: $q^\alpha = q^\alpha(u)$, $u_1 \leq u \leq u_2$ where P_1: $q^\alpha(u_1)$ and P_2: $q^\alpha(u_2)$, and then

$$ds = \sqrt{g_{\alpha\beta} \; q'\alpha q'\beta} \quad du,$$

where $g'^\alpha = dq^\alpha/du$.

This permits the integral (V-23) to be written in the form

$$I = \int_{s_1}^{s_2} G[S_0 - \nu][f(S_0 - \nu]^2 \frac{ds}{[f(S_0 - \nu)]}$$

$$= \int_{u_1}^{u_2} G[S_0 - \nu] f(S_0 - \nu) \sqrt{g_{\alpha\beta} \frac{dq^\alpha}{du} \frac{dq^\beta}{du}} \quad du \qquad \text{(V-24)}$$

or

$$I = \int_{u_1}^{u_2} \sqrt{H(S_0 - \nu) \;\; g_{\alpha\beta} \frac{dq^\alpha}{du} \frac{dq^\beta}{du}} \; du, \qquad \text{(V-25)}$$

where

$$H(S_0 - \nu) = [G(S_0 - \nu) \;\; f(S_0 - \nu)]^2. \qquad \text{(V-26)}$$

Then the trajectories determined by this variational problem are equivalent to the geodesics in a three-dimensional Riemannian manifold with the arc element

$$ds^2 = H(S_0-\nu)\ g_{\alpha\beta}\ dq^\alpha\ dq^\beta. \tag{V-27}$$

1. Particular Integrating Factor, Zero Force F

Suppose now that the integrating factor is

$$\phi = \sqrt{1 - \frac{g_{\alpha\beta}\ \dot{q}^\alpha\ \dot{q}^\beta}{c^2}}\ .$$

Then the action integral, equation (V-15) may be written

$$I = \int_{t(p_1)}^{t(p_2)} \frac{m\ g_{\alpha\beta}\ \dot{q}^\alpha\ \dot{q}^\beta}{\sqrt{1 - \frac{g_{\alpha\beta}\ \dot{q}^\alpha \dot{q}^\beta}{c^2}}}\ dt. \tag{V-28}$$

Now,

$$\tau = mc^2 [1 - \sqrt{1 - \frac{g_{\alpha\beta}\ \dot{q}^\alpha\ \dot{q}^\beta}{c^2}}\ ]\ ,$$

or

$$(1 - \frac{\tau}{mc^2})^2 = 1 - \frac{g_{\alpha\beta}\ \dot{q}^\alpha\ \dot{q}^\beta}{c^2}\ ;$$

so that

$$(\frac{ds}{dt})^2 = g_{\alpha\beta}\,\dot{q}^{\alpha}\,\dot{q}^{\beta} = c^2[1 - (1 - \tau/mc^2)^2]. \qquad \text{(V-29)}$$

Substituting (V-29) into (V-28) yields

$$I = \int_{t(p_1)}^{t(p_2)} \frac{mc^2[1 - (1 - \tau/mc^2)^2]dt}{\sqrt{1 - 1 + (1 - \tau/mc^2)^2}},$$

or

$$I = \int_{t(p_1)}^{t(p_2)} \frac{mc^2[1 - (1 - \tau/mc^2)^2]dt}{I - \tau/mc^2}. \qquad \text{(V-30)}$$

Since on C' $\tau + \nu - S_0 = 0$, then $\tau = S_0 - \nu$ and

$$(\frac{ds}{dt})^2 = c^2[1 - (1 - \tau/mc^2)^2],$$

or

$$dt = \frac{ds}{\sqrt{c^2[1 - (1 - \frac{S_0 - \nu}{mc^2})^2]}}. \qquad \text{(V-31)}$$

Putting equation (V-31) into equation (V-30)

$$I = \int_{s_1}^{s_2} \frac{mc^2[1 - (1 - \frac{S_0 - \nu}{mc^2})^2]\,ds}{(1 - \frac{S_0 - \nu}{mc^2})\sqrt{c^2[1 - (1 - \frac{S_0 - \nu}{mc^2})^2]}},$$

or

$$I = \int_{s_1}^{s_2} \frac{mc\sqrt{1 - (1 - \frac{S_0 - V}{mc^2})^2}}{(1 - \frac{S_0 - V}{mc^2})} \, ds$$

$$= \int_{s_1}^{s_2} mc \sqrt{\frac{1}{(1 - \frac{S_0 - V}{mc^2})^2} - 1} \; ds. \tag{V-32}$$

Parameterizing the varied paths by

$$ds = \sqrt{g_{\alpha\beta} \, q'^{\alpha} q'^{\beta}}$$

where $q'^{\alpha} = \frac{dq^{\alpha}}{du}$, so that on C': $q^{\alpha} = q^{\alpha}(u)$ with $u_1 \leq u \leq u_2$, then equation (V-32) becomes

$$I = \int_{u_1}^{u_2} \sqrt{\frac{1}{(1 - \frac{S_0 - V}{mc^2})^2} - 1} \sqrt{g_{\alpha\beta} \, q'^{\alpha} q'^{\beta}} \; du \tag{V-33}$$

Thus this geometrization gives an arc element

$$ds^2 = m^2 c^2 [\frac{1}{(1 - \frac{S_0 - V}{mc^2})^2} - 1] \, g_{\alpha\beta} \, dq^{\alpha} \, dq^{\beta} \tag{V-34}$$

This metric is a three-dimensional metric ($\alpha, \beta = 1,2,3$) with coefficients which are functions of position only since the $g_{\alpha\beta}$ are functions of position and $V = V(q^1, q^2, q^3)$.

If the arc element (V-34) es expanded in powers of $\left(\frac{S_0-V}{mc^2}\right)$ then

$$ds^2 = \left[2m(S_0-V) + 3\,\frac{(S_0-V)^2}{c^2} + 4\,\frac{(S_0-V)^3}{mc^4} + \ldots\right] g_{\alpha\beta}\, dq^\alpha\, dq^\beta \tag{V-35}$$

The first term of this arc element expansion is the same as the arc element in equation (B-6).

Again the metric for this space, given by equation (V-34) is consistent with the extension of the stability quadratic form for an isentropic system, namely the metric spans only the space dimensions. Thus it can be seen that the concepts of Section II can be made consistent with the dynamics of Newtonian and relativistic mechanics if isentropic systems are considered, the limiting velocity taken as the speed of light, and the entropy becomes the relativistic Hamiltonian.

D. GENERAL ISENTROPIC SYSTEM

The arc element, equation (B-6), is the classical arc element and corresponds, in the dynamic theory, to a system where it is assumed that forces $F_\alpha(q)$ exist as a function of position alone and that the forces $F_\alpha(\dot{q},q)$ are neglibible or zero. To obtain the arc element (V-34) the forces F_α were assumed to be the only forces acting on the system. These forces were functions of both position and velocity since the form of these was taken to be

$$\mathcal{F}_\alpha = \phi f_\alpha(q^1, q^2, q^3).$$

Although it is not done here, it would be desirable to find the arc element for a system subject to both types of forces. For this system

$$d\mathcal{S} = \frac{dT}{\phi} - \frac{(F_\alpha + \mathcal{F}_\alpha)}{\phi}\, dq^\alpha$$

where the forces F_α are the forces of the system and

$$V(\bar{q}) = - \int \bar{F}_\alpha \, dq^\alpha,$$

is the potential energy of the system while the forces $\mathcal{F}_\alpha$ are the forces which are inevitable. This would include the possibility that $\frac{\partial \mathcal{U}}{\partial q^\alpha} \neq 0$. For instance, if $\mathcal{U} = \frac{1}{2} m g_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta$ and the space is not Euclidean then

$$\frac{\partial \mathcal{U}}{\partial q^\gamma} = \frac{1}{2} m \, \frac{(g_{\alpha\beta})}{\partial q^\gamma} \, \dot{q}^\alpha \dot{q}^\beta \neq 0.$$

VI. NON-ISENTROPIC SYSTEM

A. FOUR-DIMENSIONAL ARC ELEMENT

1. Choosing the Arc Element

Chapter V demonstrated that it is possible for the concepts presented in Chapter II and the proposed dynamic laws to be consistent with particle dynamics. This section now returns to the point of development at which the quadratic forms of stability were attained. Up to this point the development is strictly based upon the three dynamic laws and therefore these quadratic forms reflect only the demands of these laws.

Though it is possible to arrive at more than one quadratic form which contains the stability conditions these forms would be expressed in terms of different variables. For instance, the form given by equation (IV-19) is expressed in a space-entropy manifold while the one given by equation (IV-21) is in a space-velocity manifold. Since both forms express the same requirements a choice must be made on the basis of simplicity of use, variables desired, or some other priority considerations.

Before a particular manifold is chosen it seems appropriate to recall the requirements upon a metric which ensure that the stability conditions meet these requirements. The three requirements for a metric are that the "distance"

given by the metric satisfy the following:[2]

i. $d(A,B) \geq 0$

$d(A,B) = 0$ if and only if $A = B$,

ii. $d(A,B) = d(B,A)$, and

iii. $d(A,B) \leq d(A,C) + d(C,B)$.

Thus it may be seen that these quadratic forms do define "natural" metrics for the space of the appropriate function.

Returning to the choice of a particular form, consider a metric in a space-velocity manifold. Geodesics for this manifold would be third order equations. To see this consider the quadratic form

$$d\phi^2 = A(\dot{q},q)(d\dot{q})^2 + B(\dot{q},q)(dq)^2.$$

If the arc length is used to parameterize the manifold by choosing $d\phi = v_0 dt$, then the geodesics are given by the Euler equations which makes the arc length an extremum, or

$$\delta \int_{t_1}^{t_2} \left(\frac{d\phi}{dt}\right)^2 dt = \int_{t_1}^{t_2} A(\dot{q},q)\ddot{q}^2 + B(\dot{q},q)\dot{q}^2 \; dt = 0.$$

This represents a variational problem of

$$\delta \int_{t_1}^{t_2} f(t,q,\dot{q},\ddot{q}) \; dt = 0$$

which requires a third order Euler equation.

The fact that the geodesic equations for a space-velocity manifold are third order displays the time assymmetry

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[2] Dettman, John W., <u>Mathematical Methods in Physics and Engineering</u>, p. 30, 1969.

desired. However, third order differential equations are not very nice equations to have, since there is no established method of obtaining a solution.

Now consider the quadratic form in the space-entropy manifold. This represents a simplification over the space-velocity manifold by a reduction of the order of the geodesic equations. Because of this simplification the quadratic form in the space-entropy manifold will be adopted as the metric describing the system for this section.

In order for this metric to be consistent with the previous section on isentropic systems it must reduce to the metric of the isentropic system in the event that the entropy of the system is a constant. Such a four-dimensional arc element is

$$ds^2 = \bar{h}_{ij}\, dq^i\, dq^j \;;\; (i,j = 0,\, 1,\, 2,\, 3) \tag{VI-1}$$

where $g^0 = \frac{S}{f_0}$ so that the fourth coordinate is the entropy with an appropriate scale factor, f_0, with dimensions of a force for dimensionality correctness.

Separating out the space portion of this arc element gives

$$ds^2 = \bar{h}_{00}(dq^0)^2 + 2\bar{h}_{0\alpha}dq^0\bar{d}q^\alpha + \bar{h}_{\alpha\beta}dq^\alpha dq^\beta;\ (\alpha,\beta = 1,2,3). \tag{VI-2}$$

Thus it can be seen that when the entropy is a constant the arc element reduces to

$$ds^2 = \bar{h}_{\alpha\beta}\, dq^\alpha\, dq^\beta \tag{VI-3}$$

which has the form of the arc element for the isentropic systems discussed in the previous chapter when the entropy was constant.

Then if the $\bar{h}_{\alpha\beta}$ of the isentropic system are taken as the six independent coefficients $\bar{h}_{\alpha\beta}$ the remaining four coefficients ($\bar{h}_{00}$, $\bar{h}_{0\alpha}$) in the four-dimensional arc element must be found as functions of coordinates and entropy for non-isentropic systems. If the ten independent coefficients of the arc element (VI-3) are determined then this arc element is a general arc element for this dynamic theory. But this arc element is not the only choice that could be made. However, the choice of the four-dimensional arc element (VI-1) provides the arc element with the fewest independent variables which will ensure that the stability conditions are met.

2. Parameterization

Thus far in the discussion the variable t has appeared in the notion of velocity by specifying velocity as a function of t and as a parameter in the equations of motion. The manner in which t has been used gave it the same absolute quality as time in Newtonian mechanics and it may be defined and measured in any appropriate manner. In the second dynamical law time and space are coupled through the integrating factor in terms of the absolute velocity.

In a geodesic approach to dynamics it is convenient to parameterize the space in a particular manner. For instance, if the metric properties of the manifold are determined by

$$ds^2 = \bar{h}_{ij}\, dq^i\, dq^j \; ; \quad (i,j = 0,1,2,3)$$

the length of the curve C, represented in R_4 by equation C: $q^i = q^i(t)$, $t_1 \leq t \leq t_2$, is given by

$$s = \int_{t_1}^{t_2} \sqrt{\bar{h}_{ij}\, \dot{q}^i\, \dot{q}^j}\; dt. \tag{VI-4}$$

The element of the functional (VI-4) are the geodesics in R_4. Since $\frac{ds}{dt} = \sqrt{\bar{h}_{ij}\, \dot{q}^i\, \dot{q}^j}$, carrying out the indicated differentiation required by Euler's equations to determine an extremum of the functional (VI-4) results in the equations,[5]

$$\bar{h}_{ij}\, \ddot{q}^i + [ik,j]\, \dot{q}^i\, \dot{q}^k = \bar{h}_{ij}\, \dot{q}^i\, \frac{d^2s/dt^2}{ds/dt}\,, \tag{VI-5}$$

as the desired equations of geodesics. These equations may be simplified by a choice of the parameter t that sets the right hand side equal to zero. The choice of the parameterization which does this is

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[5]Sokolnikoff, I.S., Tensor Analysis Theory and Applications to Geometry and Mechanics of Continua, p. 158, 1964.

$$\frac{ds}{dt} = \sqrt{\bar{h}_{ij}\,\dot{q}^i\,\dot{q}^j} = \text{constant} = \dot{s}_0$$

which has the dimensions of a velocity.

In the thermodynamic development of the second law an absolute velocity was defined. If the manifold is parameterized using this velocity the results will yield a time consistently defined by the laws of dynamics. Therefore, if this absolute velocity is defined as c, the space may be parameterized by[6]

$$\frac{ds}{dt} = \sqrt{\bar{h}_{ij}\,\dot{q}^i\,\dot{q}^j} = c. \tag{VI-6}$$

B. EQUATIONS OF MOTION

1. Square of Momentum

Just as in classical mechanics, several forms of equations of motion are possible, therefore, it may be beneficial to present several different approaches here in order to help interpret the metric coefficients. One approach would be to empirically determine the metric coefficients which seem to correspond to reality, while another would be to seek equations or relations that the coefficients must satisfy.

The limitation imposed by the number of symbols available requires a comment on notation. First, the metric coefficients corresponding to spatial coordinates alone will be denoted by $g_{\alpha\beta}$; $(\alpha, \beta = 1,2,3)$ as previously

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[6] See Appendix F.

used and are determined by the choice of spatial coordinate system (i.e. rectangular, cylindrical, etc.). The metric coefficients h_{ij} will be used to denote the coefficients in the Riemannian space determined by geometrizing the dynamic system. Latin indices will be used when the indices may assume any of the four values 0,1,2,3 while Greek indices may assume only the values 1,2,3. The coefficients $\bar{h}_{ij}$ correspond to the arc element $(ds)^2$, as the $g_{\alpha\beta}$ corresponded to the arc element $(ds)^2$ in the isentropic systems. The h_{ij} are the coefficients of the arc element $(dS)^2$ which are the potential functions which geometrize the space, as the $h_{\alpha\beta} = 2m(h-V)g_{\alpha\beta}$ did in the isentropic systems, see Table 2. Concepts such as momentum, "Lagrangian", and "Hamiltonian" in the four dimensional manifold will be denoted by $\tilde{P}_i$, $\tilde{L}$, or $\tilde{H}$ to avoid confusion with their three-dimensional definitions. Note the manifold considered here is three-space with entropy as the fourth dimension.

COEFFICIENTS	$(ds)^2$	$(dS)^2$
(3 dimensional) Isentropic	$g_{\alpha\beta}$	$h_{\alpha\beta} = 2m(h-V)g_{\alpha\beta}$
(4 dimensional) Non-isentropic	$\bar{h}_{ij}$	h_{ij}

Table 2. Metric Coefficients

Now consider the definition, with previously chosen parameterization

$$\tilde{L} \equiv mc^2 \equiv m\left(\frac{ds}{dt}\right)^2 = m\, h_{ij}\, \dot{q}^i\, \dot{q}^j \; ; \quad (i,j = 0,1,2,3) \qquad \text{(VI-7)}$$

and define the four-dimensional canonical momentum

$$\tilde{P}_i \equiv \frac{\partial \tilde{L}}{\partial \dot{q}^i} = m\, h_{ij}\, \dot{q}^j \qquad \text{(VI-8)}$$

Then the contravariant four-dimensional canonical momentum is given by

$$\tilde{P}^i = h^{ij}\, \tilde{P}_j = h^{ij}\, m\, h_{jk}\, \dot{q}^k \qquad \text{(VI-9)}$$

so that

$$\tilde{P}^i \tilde{P}_{\dot{q}} = h^{ij} m h_{jk} \dot{q}^k m h_{i\ell} \dot{q}^\ell = m^2 \delta_{ik} \dot{q}^k h_{i\ell} \dot{q}^\ell$$

$$= m(m h_{i\ell} \dot{q}^i \dot{q}^\ell)$$

but

$$mc^2 = m h_{i\ell} \dot{q}^i \dot{q}^\ell$$

therefore

$$\tilde{P}^i \tilde{P}_j = m^2 c^2 \qquad \text{(VI-10)}$$

Equation (VI-10) should play an analogous role in the R_4 manifold as equation (V-13) does in the three-dimensional space of an isentropic system.

2. Lagrangian and Hamiltonian Equations

Again consider the definitions (VI-7) and (VI-8) and the additional definition

$$\tilde{F}_i \equiv \frac{\partial \tilde{L}}{\partial q^i} = m \frac{\partial h_{jk}}{\partial q^i} \dot{q}^j \dot{q}^k = \dot{q}^j \frac{\partial \tilde{P}_j}{\partial q^i} \qquad \text{(VI-11)}$$

The equations of geodesics are then

$$\frac{d}{dt} [\frac{\partial \tilde{L}}{\partial \dot{q}^i}] - \frac{\partial \tilde{L}}{\partial q^i} = 0 \qquad \text{(VI-12)}$$

or

$$\frac{d}{dt} [\tilde{P}_i] = \tilde{F}_i \qquad \text{(VI-13)}$$

Equation (VI-12) may be written as

$$\frac{d}{dt} [\tilde{L} - \dot{q}^i \tilde{P}_i] = 0$$

so that integration yields

$$\tilde{L} - \dot{q}^i \tilde{P}_i = \text{constant.}$$

Again define this constant as

$$\tilde{H} \equiv \dot{q}^i \tilde{P}_i - \tilde{L}, \qquad \text{(VI-14)}$$

and write $\tilde{L}$ as

$$\tilde{L} = \int \tilde{P}_i \, d\dot{q}^i + \int \tilde{F}_i \, dq^i .$$

Then $\tilde{H}$, where the integrals are taken along the trajectory, is given by

$$\tilde{H} = \dot{q}^i \tilde{P}_i - \int \tilde{P}_i \, d\dot{q}^i - \int \tilde{F}_i \, dq^i . \qquad \text{(VI-15)}$$

The first two terms of equation (VI-15) may be integrated by parts so that $\tilde{H}$ becomes

$$\tilde{H} = \int \dot{q}^i \, d\tilde{P}_i - \int \tilde{F}_i \, dq^i. \qquad \text{(VI-16)}$$

Differentiating equation (VI-16) and recalling equation (VI-13) the Hamiltonian form of the equations of motion may be written as

$$\frac{\partial \tilde{H}}{\partial \tilde{P}_i} = \dot{q}^i \; ; \qquad \frac{\partial \tilde{H}}{\partial q^i} = - \tilde{F}_i = - \frac{d\tilde{P}_i}{dt} ,$$

or

$$\frac{\partial \tilde{H}}{\partial \tilde{P}_i} = \dot{q}^i \; ; \qquad \frac{\partial \tilde{H}}{\partial q^i} = - \frac{d\tilde{\phi}_i}{dt} \qquad \text{(VI-17)}$$

The form of the equations of motion remain the same as the three-dimensional form and therefore reduce to the same equations when the entropy is a constant and the $h_{\alpha\beta}$ are the $h_{\alpha\beta}$ for an isentropic system.

3. Principle of Least "Action"

Geometrization, for the isentropic system, was achieved by considering a change of variables in the principle of least action which converted the variational problem from one with variable end points to a problem with fixed end points. In order to help interpret the ten coefficients in the four-dimensional arc element for the non-isentropic system the same approach may be followed.

In the isentropic system the principle of least action involved the functional

$$A = \int_{t(p_1)}^{t(p_2)} 2\,T\,dt$$

where

$$T \equiv \frac{1}{2}\, m\, g_{\alpha\beta} \frac{dq^{\alpha}}{dt} \frac{dq^{\beta}}{dt}\ ; \quad (\alpha,\beta = 1,2,3)$$

and the $g_{\alpha\beta}$ were to be determined by the choice of coordinate system.

For the non-isentropic system the analogous functional would be

$$A = \int_{t(p_1)}^{t(p_2)} 2\,\tilde{T}\,dt \qquad \text{(VI-20)}$$

where

$$\tilde{T} \equiv \frac{1}{2}\, m\, \bar{h}_{ij}\, \frac{dq^i}{dt}\, \frac{dq^j}{dt}\ ; \ (i,j = 0,1,2,3) \qquad \text{(VI-21)}$$

and the $\bar{h}_{ij}$ are to be determined by the choice of coordinate system. In the event that the entropy is a constant then this functional should reduce to the functional for the isentropic system, therefore the six coefficients $\bar{h}_{\alpha\beta}$ must satisfy the relations

$$\bar{h}_{\alpha\beta} = g_{\alpha\beta}$$

The statement "the $\bar{h}_{ij}$ are to be determined by the choice of coordinate system" raises a question about whether or not the freedom of choice exists. It seems appropriate here to discuss the types of geometric theories. Two types of geometric theories are:

i) Theories with absolute elements: In these theories the geometry is predetermined. The events and the dynamical laws are embedded into this geometrical framework. The metric represents "absolute elements" injected into the theory.

ii) True geometric theories: Here the metric itself becomes the dynamic element and is determined by certain dynamic laws, as in Einstein's theory.

The approach here is i) as far as $\bar{h}_{ij}$ is concerned, but the h_{ij} must be determined by physical laws. Thus the choice of the $\bar{h}_{ij}$ may be made but the h_{ij} are to be determined.

The constraint associated with the variational problem in the isentropic system case, which made the resulting Euler's equations equivalent to the Lagrangian equations of motion, was that the Hamiltonian was a constant. For the non-isentropic system this constant is given by the equations (VI-14) and (VI-16) or

$$\tilde{H} = \int \dot{q}^i \, d\tilde{P}_i - \int \tilde{F}_i \, dq^i \qquad \text{(VI-22)}$$

where $\tilde{P}_i$ was defined to be

$$\tilde{P}_i \equiv m \, h_{ij} \, \dot{q}^j \; .$$

Then the statement of the principle of least action for the non-isentropic (four-dimensional) system becomes:

> Of all curves C' passing through P_1 and P_2 in the neighborhood of the trajectory C, which are traversed at a rate such that, for each C', for every value of t, $\tilde{H} - \tilde{H}_0 = 0$, that one for which the action integral A (equation (VI-20)) is stationary is the trajectory of the particle.

Then if the following definitions are made

$$\tilde{K}(q^0,\ldots,q^3,\dot{q}^0,\ldots,\dot{q}^3) \equiv \int \dot{q}^i \, d\tilde{P}_i$$

$$\tilde{V}(q^0,\ldots,q^3) \equiv -\int \tilde{F}_i \, dq^i, \tag{VI-23}$$

the function $\tilde{H}$ may be written as

$$\tilde{H} = \tilde{K} + \tilde{V}. \tag{VI-24}$$

If $\tilde{H} - \tilde{H}_0 = 0$ so that

$$K + V - \tilde{H}_0 = 0, \tag{VI-25}$$

in principle, the first term in equation (VI-25) may be solved for the four-dimensional velocity

$$\left(\frac{ds}{dt}\right)^2 \equiv \bar{h}_{ij} \frac{dq^i}{dt} \frac{dq^j}{dt} = \frac{2T}{m} \tag{VI-26}$$

as a function of K or

$$\left(\frac{ds}{dt}\right)^2 = [f(K)]^2$$

But $K = \tilde{H}_0 - V$ then

$$\bar{h}_{ij} \frac{dq^i}{dt} \frac{dq^j}{dt} = [f(\tilde{H}_0 - V)]^2 \tag{VI-27}$$

Substituting equations (VI-26) and (VI-27) into the integral (VI-20) yields

$$A = \int_{s_1}^{s_2} \frac{m[f(\tilde{H}_0 - V)]^2 \, ds}{f(\tilde{H}_0 - V)}$$

or

$$A = \int_{s_1}^{s_2} mf(\tilde{H}_0 - V) \, ds \; . \qquad \text{(VI-28)}$$

The varied paths can be parameterized so that C: $q^i = q^i(u)$, $u_1 \leq u \leq u_2$ where P_1: $q^i(u_1)$ and P_2: $q^i(u_2)$, then

$$ds = \sqrt{\bar{h}_{ij} \, q'^i q'^j} \, du$$

where

$$q'^i = \frac{dq^i}{du} \; .$$

Then equation (VI-28) becomes

$$A = \int_{u_1}^{u_2} mf(\tilde{H}_0 - V)\sqrt{\bar{h}_{ij} \, q'^i \, q'^j} \quad du$$

or

$$A = \int_{u_1}^{u_2} \sqrt{H(\tilde{H}_0 - V)\, \bar{h}_{ij}\, q'^i\, q'^j}\; du \qquad \text{(VI-29)}$$

where

$$H(\tilde{H}_0 - V) = [mf(\tilde{H}_0 - V)]^2.$$

The trajectories determined by this variational problem are equivalent to the geodesics in a four-dimensional Riemannian manifold with the arc element

$$(dS)^2 = h_{ij}\, dq^i\, dq^j \qquad \text{(VI-30)}$$

where

$$h_{ij} = H(\tilde{H}_0 - V)\, \bar{h}_{ij}.$$

If the entropy is a constant then this arc element should reduce to the arc element of the isentropic system so that the six coefficients $h_{\alpha\beta}$ must reduce to

$$h_{\alpha\beta}\Big|_{\dot{q}=\dot{q}_0} = H[\tilde{H}_0 - V(\dot{q}_0, q^1, q^2, q^3)] = \begin{cases} 2m(h - V(q^1,q^2,q^3))\, g_{\alpha\beta}; \\ \quad \text{if } V \neq 0;\ V = 0, \\ \\ m^2c^2 \quad \dfrac{1}{\left(1 - \dfrac{S_0 - V}{mc^2}\right)^2} - 1\ g_{\alpha\beta} \\ \\ \quad \text{if } V = 0;\ V \neq 0 \text{ and} \\ \quad \phi = \sqrt{1 - u^2/c^2}\ . \end{cases}$$

The equations of motion presented here describes the motion provided the coefficients in the arc element are known. It is at this point where the abstract formulism must yield to empirical facts. For if these equations are to describe a real system the equation of state must be known for the system. In thermodynamics an equation of state may have a form such as PV = nRT. For Newtonian mechanics force laws are needed. Electrodynamics obtain the force laws from Maxwell equations which contain the empirical facts. General relativity offers a system of differential equations which may be solved to find the metric coefficients in a space-time manifold. The determination of the coefficients is not addressed here, however, in Appendix F the manner in which some of the coefficients appear in forces may be seen.

The space-entropy manifold with its equations does not readily display consistency with Newtonian and relativistic dynamics. Section V demonstrated the desired consistency for an isentropic, Q-conservative system. But the procedure of Section V does not show that Newtonian and relativistic dynamics can be logically derived from the three proposed dynamical laws. To show that they do indeed follow from the three laws consider a Q-conservative system described in the space-entropy manifold. For this system the mechanical entropy principle must hold. Appendix D shows that this system is governed by an arc element in a space-time manifold which becomes the Minkowski space of relativistic

dynamics when the space-entropy manifold is Euclidean. Thus relativistic dynamics, and hence, in the low velocity limit, Newtonian dynamics, follows from the application of the three dynamical laws for a Q-conservative system.

VII. CONCLUSIONS

This investigation was motivated by a number of questions which were presented in the introduction. Some answers provided by the results of the investigation can now be stated:

a. The first question was whether or not the speed of light was the only characteristic velocity in nature. The answer is provided by the axiomatic development of the second law. The axiomatic development produced an integrating factor for the differential statement of the first law. A characteristic velocity was shown to exist in the definition of the absolute velocity. That absolute velocity is given by a constant velocity process at which the integrating factor is zero. The important point in the development which provides the answer to the uniqueness of this velocity is the proof that the integrating factor is independent of the nature of the force. Therefore if the absolute velocity is independent of the force it must be applicable to all forces and hence unique.

Since by definition the absolute velocity is a constant in one reference frame it must also be a constant in any other reference frame moving with a constant velocity relative to the first. Thus the absolute velocity must be unique and a constant in all reference frames moving with constant relative velocities. The experimental and

theoretical evidence of electromagnetism requires that this absolute velocity be the same value in all these reference frames. This requirement leads to the principle of Lorentz covariance. Then all laws of nature must be Lorentz covariant whether electromagnetic, gravitational or weak interactions since the absolute velocity is unique and independent of the force.

b. The second question was whether all dynamics should share time assymmetry. The question of "should" is not answered here. The formulation has introduced a directivity into the dynamics as evidenced by equation (II-15) which is the mechanical equivalent of the thermodynamic principle of increasing entropy. This principle is the basis for the qualitative prediction of expanding planetary orbits in Appendix E.

c. Another question which motivated this investigation but was not presented in the introduction involves effective mass as a function of velocity and/or force as a function of velocity. Suppose that the theory of relativity had not yet been proposed so that Newton's dynamical equations were not yet required to be Lorentz covariant. Then suppose that an experiment were conducted which required the introduction of a velocity dependence into Newton's second law in order to describe the motion. What would be the difference in assuming that the force was velocity dependent or assuming an effective mass that was velocity dependent?" Then the question is whether or not the proper modification of Newton's second

law may be obtained by an approach other than by the special theory of relativity. This question is answered in Section V where it was shown that the integrating factor $\phi = \sqrt{1 - \dot{q}^2/c^2}$ yields relativistic equations.

This formulization of dynamics allows some further conclusions to be drawn about the velocity dependence of mass and force. The differential expression for the energy exchange from Section V is

$$\bar{d}Q = m\dot{q}d\dot{q} - \sqrt{1 - \dot{q}^2/c^2}\; F(q)dq$$

while the differential expression for the entropy was

$$d\mathcal{S} = \frac{m\dot{q}d\dot{q}}{\sqrt{1 - \dot{q}^2/c^2}} - F(q)\; dq.$$

The expression for $\bar{d}Q$ is the statement of the first law, however the integral of this expression is dependent upon the path. The expression for $d\mathcal{S}$ is a modification of the first law whose integral is path independent. Both expressions may be considered as representing the systems' dynamics. If $\bar{d}Q$ is considered the "real" energy transfer then $d\mathcal{S}$ might be interpreted as the "effective" energy transfer. Following this interpretation then m is the "real" mass and $\sqrt{1 - \dot{q}^2/c^2}\; F(q)$ is the "real" force while $\frac{m}{\sqrt{1 - \dot{q}^2/c^2}}$ and F(q) becomes the "effective" mass and "effective" force respectively. In this interpretation the "real" mass

is independent of the velocity and the "real" force is velocity dependent. However the difficulty of working with a path dependent integral can be avoided by using the "effective" differential expression with "effective" mass and force. This represents a change in viewpoint but not necessarily a change in the mathematical expression used. This can be seen by considering that in special relativistic mechanics only motion for which the relativistic Hamiltonian remains constant is considered. In the theory given here, motion for which the entropy, whose mathematical form is the same as the relativistic Hamiltonian, is a constant, represents only a special case of all possible motion, namely isentropic motion.

It is not possible to say that this investigation supports the conclusion that the laws formulated here produce equations which contain all existing dynamical theories. One reason is that a quantum description was not even mentioned. However, in Chapter V consistency with the special theory of relativity was displayed in the equations obtained for the case of an isentropic system. Consistency with Newtonian mechanics was shown for an isentropic system as the low velocity limit of the relativistic equations. It is more difficult to determine the consistency between this theory and General Relativity theory though Appendix D gives some indication of their relationship.

Appendix D contains the derivation of the Minkowski space-time arc element for a Q-conservative system. Thus it may be concluded that relativistic dynamics does indeed follow from the three dynamical laws when Q-conservative systems are considered.

Further theoretical and experimental investigation is necessary before a definite conclusion can be made about the prediction of expanding orbits. Several questions must be answered, such as: is it possible to find an expression of orbital motion allowing both a rotation of perihelion and a change in the semi-major axis which would represent an approximation to the solution of the equations of motion, how closely does planetary motion approximate an isolated system, is the motion really irreversible, etc.?

As any newly proposed theory which is offered to answer a particular question, this proposed formulation of dynamics leads to numerous new questions. Some of these questions could be: What does the principle of increasing entropy mean? Might not the planets be in slowly increasing orbits as a result of following irreversible trajectories? Could this irreversibility (directivity) be the origin of the expansion of the universe? From equation (D-10) in Appendix D it can be seen that for a Euclidean space

$$d\tau = dt\sqrt{1 - v^2/c^2} \quad ,$$

where $dq^0 = \frac{cd\tau}{h_{00}}$, is it not possible then to interpret the entropy as a measure of "time" for the system? Would this not lead to the interpretation that the principle of increasing entropy requires that a system evolve in "time" or get "older"?

APPENDIX A

EQUIVALENCE OF THE TRANSFORMATION STATEMENTS OF THE SECOND LAW

This appendix provides the proof of the equivalence of the two transformation statements of the second law and the development necessary for stating the generalized "Carnot" theorem for mechanical systems.

The two transformation statements are restated here as:

I. There exists no dynamic transformation whose sole effect is to extract a quantity of energy from a given reservoir (or source) and to convert it entirely into work.

II. There exists no dynamic transformation whose sole effect is to extract a quantity of energy from a reservoir while the system is at one velocity and deliver this energy to another reservoir while at a higher velocity.

To show the equivalence of these two statements, first assume I is false and show II must be false, then reverse the roles.

Suppose I is false. Then energy may be extracted from a reservoir while the system is at a velocity $\dot{q}_1$ and converted entirely into work, with no other effect. This work can then be converted into energy and delivered to a reservoir while the system is at $\dot{q}_2 > \dot{q}_1$ with no other effect. The net result of this two-step process is the transfer of

energy obtained by the system from one reservoir while at one velocity to another reservoir while at a higher velocity with no other effect. Hence II is false.

Here the importance of the words "sole effect" and "no other effect" becomes more visible. For example, an electron cannot absorb energy from an electric field, thereby increasing its velocity, then pass that acquired energy to another reservoir through collision or some other means without radiating. The radiation is then the "other effect."

To complete the proof of the equivalence to the two statements of the second law, first define an "engine" to be a system that can undergo a cyclic transformation in which the system does the following things, and only the following things:

a. absorbs an amount of energy $Q_2 > 0$ while at $\dot{q}_2$;
b. rejects an amount of energy $Q_1 > 0$ while at $\dot{q}_1$, with $\dot{q}_1 < \dot{q}_2$;
c. performs an amount of work $W > 0$.

Now suppose II is false. Extract Q_2 at $\dot{q}_1$ and reject it at $\dot{q}_2$, with $\dot{q}_2 > \dot{q}_1$. Operate an engine between $\dot{q}_2$ and $\dot{q}_1$ for one cycle, and arrange the engine so that the amount of energy extracted by the engine at $\dot{q}_2$ is exactly Q_2. The net result is that an amount of energy is extracted at $\dot{q}_1$ and entirely converted into work, with no other effect. Hence I is false. Therefore the statements are equivalent.

With this statement of the second law a special reversible process called a mechanical "Carnot" engine may be defined. A Carnot engine is one that makes a complete cyclic transformation in a completely reversible way. The cyclic process of a Carnot engine is illustrated in Figure A1 where ab is a constant velocity process at velocity $\dot{q}_2$, during which the system absorbs energy Q_2: bc is conservative; cd is a constant velocity process at velocity $\dot{q}_1$, with $\dot{q}_1 < \dot{q}_2$, during which the system rejects energy Q_1; and da is conservative. The work done by the system in one cycle is, according to the first law,

$$\mathcal{W} = Q_2 - Q_1$$

since $\Delta\mathcal{U} = 0$ in any cyclic transformation. The efficiency of the engine is defined to be

$$\mathcal{N} \equiv \frac{\mathcal{W}}{Q_2} = 1 - \frac{Q_1}{Q_2} .$$

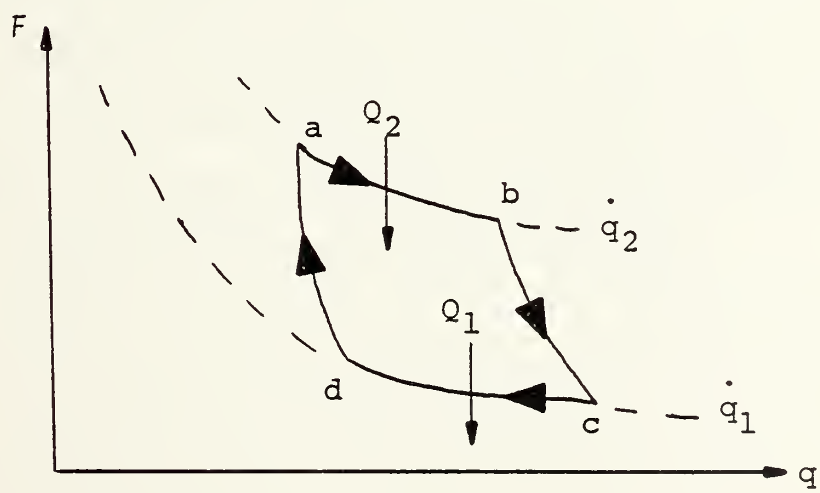

FIGURE A1. Carnot Engine

The conclusion can then be made that, if $W > 0$, then $Q_1 > 0$ and $Q_2 > 0$. This conclusion may be reached as follows. Q_1 must not be zero for if it were the system would be capable of absorbing the energy Q_2 and converting this energy completely into work which is a violation of statement I. Suppose $Q_1 < 0$. This means that the engine absorbs the amount of energy Q_2 at velocity $\dot{q}_2$ and the amount of energy $-Q_1$ at $\dot{q}_1$ and converts the net amount of energy $Q_2 - Q_1$ into work. This amount of work, which by assumption is positive, may be converted into energy and delivered to the reservoir at $\dot{q}_2$, with no other effect. The net result is the transfer of the positive amount of energy $-Q_1$ from $\dot{q}_1$ to $\dot{q}_2$ with no other effect. Since $\dot{q}_2 > \dot{q}_1$ by assumption, this is impossible by statement II. Therefore $Q_1 > 0$. From $W = Q_2 - Q_1$ and $W > 0$ it follows that $Q_2 > 0$.

The same procedure can be used to show that if $W < 0$, then $Q_1 < 0$ and $Q_2 < 0$.

Then a generalized Carnot theorem may be proven but is stated here without proof:

Theorem: No process operating between two given velocities is more efficient than a Carnot process.

Corollary: All Carnot processes operating between two velocities have the same efficiency.

APPENDIX B

CLASSICAL GEOMETRIZATION

The geometrization of classical dynamics is provided by the principle of least action. Therefore a review of this principle may prove beneficial in the geometrization of dynamics governed by the three dynamic laws.

The principle of least action: Of all curves C' passing through P_1 and P_2 in the neighborhood of the trajectory C, which are traversed at a rate such that, for each C', for every value of t, T + V = h, that one for which the action A is stationary is the trajectory of the particle.

When stated in the form of the variational equation, this principle reads

$$\delta \int_{t(p_1)}^{t(p_2)} 2\ T\ dt = 0, \tag{B-1}$$

with the auxillary condition

$$T + V - E_0 = 0, \tag{B-2}$$

where h is a constant.

It is important to recognize that in this instance the extremals of the action integral cannot be determined by setting the function in Euler's equations equal to 2T because of the auxillary condition. Since T is a function

of the velocity v, and V is a function of position x alone, the times $t(P_2) - t(P_1)$ required to traverse the varied paths C' will differ in general. Thus the upper limit in the integral (B-1) is not fixed. One approach to the solution is to consider a change of variables. Since the kinetic energy

$$T = \frac{m}{2} g_{\alpha\beta} \frac{dx^{\alpha}}{dt} \frac{dx^{\beta}}{dt} = \frac{m}{2} \left(\frac{ds}{dt}\right)^2 ,$$

$$dt = \sqrt{\frac{m}{2\,T}} \quad ds,$$

$$= \sqrt{\frac{m}{2(E_0 - V)}} \quad ds. \tag{B-3}$$

Consequently the action integral can be written

$$A = \int_{s_1}^{s_2} \sqrt{2m(E_0 - V)} \quad ds, \tag{B-4}$$

since along all admissable paths $T = E_0 - V$. The integrand in the preceding integral is clearly independent of t. The varied paths can be parametrized so that C: $x^{\alpha} = x^{\alpha}(u)$, $u_1 \leq u \leq u_2$, where P_1: $x^{\alpha}(u_1)$ and P_2: $x^{\alpha}(u_2)$, and write

$$ds = \sqrt{g_{\alpha\beta}\ x'^{\alpha}\ x'^{\beta}} \quad du,$$

where

$$x'^{\alpha} = \frac{dx^{\alpha}}{du}.$$

This permits the action integral to be written in the form

$$A = \int_{u_1}^{u_2} \sqrt{2m(E_0 - V)\, g_{\alpha\beta}\, x'^{\alpha}\, x'^{\beta}}\; du, \tag{B-5}$$

and since the limits of integration in (B-5) are fixed, the determination of the trajectories is equivalent to finding geodesics in a three-dimensional Riemannian manifold with arc element

$$ds^2 = 2m(E_0 - V)\, g_{\alpha\beta}\, dx^{\alpha}\, dx^{\beta}\ . \tag{B-6}$$

The Euler equations may be formed so that

$$G_{x^{\alpha}} - \frac{d}{du}\,(G_{x'^{\alpha}}) = 0,$$

with

$$G = \sqrt{2m(E_0 - V)\, g_{\alpha\beta}\, x'^{\alpha}\, x'^{\beta}}\ ,$$

and recall that

$$dt = du\sqrt{\frac{mg_{\alpha\beta}x'^{\alpha}x'^{\beta}}{2(h - V)}}$$

which leads to the Lagrangian equations of motion

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^{\alpha}}\right) - \frac{\partial T}{\partial x^{\alpha}} = - \frac{\partial V}{\partial x^{\alpha}}, \quad (\alpha = 1,2,3).$$

APPENDIX C

Integrating Factor for n Dimensions

In the development of the integrating factor the discussion was limited to a one-dimensional system for simplicity. It now becomes necessary to consider the extension to systems with greater dimensionality, particularly the three-dimensional space of classical dynamics.

In one dimension the differential of the entropy was written as

$$dS = \frac{\bar{d}Q}{\phi} = f(\sigma)\ d\sigma .$$

Then if for each dimension the exchange of energy is denoted to be $\bar{d}Q_i$, then

$$dS_i = \frac{\bar{d}Q_i}{\phi_i} = f_i\ d\sigma_i ,$$

where there is no summation intended for $f_i d\sigma_i$. Since each dS_i is a perfect differential then the total change in mechanical entropy may be written as

$$dS = \sum_i dS_i = \sum_i \frac{\bar{d}Q_i}{\phi_i} = \sum_i f_i d\sigma_i .$$

However, the question which arises is whether there exists a single integrating factor ϕ such that

$$dS = \frac{\bar{d}Q}{\phi} = \sum_i \frac{\bar{d}Q_i}{\phi_i} = \sum_i f_i d\sigma_i$$

To see this consider the element of work considered before as

$$\bar{d}W = \sum_i F_i \, dq^i \; ; \quad i = 1, \ldots, n.$$

Since each dU_i is in itself a perfect differential then $dU = \sum_i dU_i$ so that

$$\bar{d}Q = \sum_i dU_i - \sum_i F_i dq^i = \sum_i (dU_i - F_i dq^i)$$

or

$$\bar{d}Q = \sum_i \bar{d}Q_i .$$

If the system is total Q-conservative in the sense that

$$\bar{d}Q = \sum_i \bar{d}Q_i = 0,$$

then $\bar{d}Q = 0$ is a Pfiffian differential equation. This equation is integrable and has an integrating factor ϕ. The integrability is guaranteed by the dynamical second law since it is impossible to go from one initial state to <u>any</u> neighboring state. Then, just as in the one-dimensional case, the perfect differential follows

$$dS = \frac{\bar{d}Q}{\phi} = \frac{\sum_i \bar{d}Q_i}{\phi}$$

But since

$$\bar{d}Q_i = \phi_i \; f_i \; d\sigma_i$$

then

$$dS = \sum_i \frac{\phi_i \; f_i \; d\sigma_i}{\phi} \; .$$

Now following the same argument presented in Section II concerning the composite system,

$$\bar{d}Q = \lambda \; d\sigma$$

where σ is a function of all the σ_i and the $\dot{q}^i$. Therefore since

$$\bar{d}Q_i = \lambda_i \; d\sigma_i$$

then

$$\bar{d}Q = \sum_i \lambda_i \{\frac{\partial \phi_i}{\partial \sigma} \, d\sigma + d\sigma_i\} \; .$$

Now

$$d\sigma = \sum_i \left(\frac{\partial\phi}{\partial\dot{q}^i}\, d\dot{q}^i + \frac{\partial\sigma}{\partial\sigma_i}\, d\sigma_i\right)$$

so that

$$\bar{d}Q = \sum_i \bar{d}Q_i$$

or

$$\lambda\, d\sigma = \sum_i \lambda_i\, d\sigma_i$$

and

$$d\sigma = \sum_i \frac{\lambda_i}{\lambda}\, d\sigma_i \ .$$

It follows then that the $\frac{\partial\sigma}{\partial\dot{q}^i} = 0$ and that the ratios λ_i/λ are also independent of the $\dot{q}^i$. Therefore the λ's have the form

$$\lambda_i = \phi\, f_i$$

$$\lambda = \phi\, F(\sigma_1, \sigma_2, \ldots, \sigma_n)$$

and also

$$\frac{\bar{d}Q}{\phi} = F\, d\sigma = \sum_i F\, \frac{\lambda_i}{\lambda}\, d\sigma_i = \sum_i \frac{\lambda_i d\sigma_i}{\phi}$$

$$= \sum_i f_i\, d\sigma_i .$$

The right hand side is a perfect differential and therefore so is the left.

Since λ_i/f_i is an integrating factor and λ/F is also an integrating factor it follows that $\phi(\dot{q}^1, \dot{q}^2, \ldots, \dot{q}^n)$ is an integrating factor for the $\bar{d}Q_i$ as well as for $\bar{d}Q = \sum_i \bar{d}Q_i$. Therefore

$$dS = \frac{\bar{d}Q}{\phi} = \sum_i \frac{\bar{d}Q_i}{\phi} .$$

APPENDIX D

SPACE-TIME MANIFOLD

This investigation has not attempted transformations from one coordinate system to other coordinate systems. It is natural then to have certain reservations, or questions, about the theory that can be removed, or answered, only by a discussion within this theoretical framework of coordinate transformations. If such a discussion of transformations exposes a transformation requirement differing from the Lorentz transformations of relativistic theory then the validity and/or utility of the theory should be questioned because transformation and symmetry arguments have yielded a long list of experimentally verified theoretical predictions, especially in the fields of atomic and nuclear physics. On the other hand if Lorentz transformations are shown to be a subset of the most general transformations allowed within the theory then consideration must be given to the possibility that the theory is a more general theory with special relativity representing a portion of the theory.

Though the equations of motion obtained in Section V were shown to be consistent with the equations of the special theory of relativity, this Appendix will attempt to provide a more geometric point of view in order to better display the transformation requirements.

Again consider the four-dimensional line element of Section VI given by equation (VI-30) as

$$(dS)^2 = h_{ij}\, dq^i\, dq^j; \quad i,j = 0,1,2,3. \tag{VI-30}$$

where $dq^0 = \frac{dS}{f_0}$ is the scaled entropy. The parameterization may be chosen as discussed in Section VI-A-2 so that

$$(dS)^2 = c^2(dt)^2 = h_{00}(dq^0)^2 + h_{0\alpha}dq^0dq^\alpha + h_{\alpha\beta}dq^\alpha dq^\beta; \quad \alpha,\beta = 1,2,3 \tag{D-1}$$

when the line element is expanded in the fourth dimension. Now since the h_{ij} are not functions of dq^0 equation (D-1) may be used to find a solution of dq^0 in terms of the other parameters. To do this consider the following definitions:

$A \equiv h_{00};$ $\qquad B \equiv h_{0\alpha}\, dq^\alpha$

$D \equiv h_{\alpha\beta}\, dq^\alpha\, dq^\beta\ ;$ and $\qquad E \equiv c^2\, dt^2.$

Then equation (D-1) may be written as

$$A(dq^0)^2 + 2B(dq^0) + D = E. \tag{D-2}$$

Dividing by A and adding $(B/A)^2$ yields

$$(dq^0)^2 + 2(B/A)(dq^0) + (B/A)^2 = (\frac{E-D}{A}) + (B/A)^2,$$

from which the solution for (dq^0) may be seen to be

$$(dq^0) = -(B/A) \pm \sqrt{(\frac{E-D}{A}) + (B/A)^2} \; . \tag{D-3}$$

Squaring equation (D-3) yields

$$(dq^0)^2 = (\frac{E-D}{A}) + a(B/A)^2 \pm 2(B/A)\sqrt{(\frac{E-D}{A}) + (B/A)^2} \; ,$$

or collecting terms

$$(dq^0)^2 = (\frac{E-D}{A}) + 2(B/A)\{ (B/A) \pm \sqrt{(\frac{E-D}{A}) + (B/A)^2} \}. \tag{D-4}$$

Substituting the expressions of the defined quantities into equation (D-4) gives

$$(dq^0)^2 = \frac{c^2\, dt^2}{h_{00}} - \frac{h_{\alpha\beta}}{h_{00}}\, dq^\alpha\, dq^\beta$$

$$+ \frac{2h_{0\alpha} dq^\alpha}{h_{00}} \{\frac{h_{0\gamma}}{h_{00}} dq^\gamma \pm \sqrt{\frac{c^2 dt^2}{h_{00}} - \frac{h_{\gamma\delta} dq^\gamma dq^\delta}{h_{00}} + \frac{(h_{0\gamma} dq^\gamma)^2}{(h_{00})^2}} \}. \tag{D-5}$$

Factoring a dt out of the term in the brackets and using $\dot{q} = dq/dt$, then equation (D-5) becomes

$$(dq^0)^2 = \frac{c^2 dt^2}{h_{00}} + \frac{2h_{0\alpha}}{h_{00}}\{\frac{h_{0\gamma} \dot{q}^\gamma}{h_{00}} \pm \sqrt{\frac{c^2 dt^2}{h_{00}} - \frac{h_{\gamma\delta} \dot{q}^\gamma \dot{q}^\delta}{h_{00}} + \frac{(h_{0\gamma} \dot{q}^\gamma)^2}{(h_{00})^2}} \}.$$

$$dt\, dq^\alpha - \frac{h_{\alpha\beta}}{h_{00}}\, dq^\alpha\, dq^\beta. \tag{D-6}$$

One result of the second law was that a Q-conservative system in equilibrium was at a maximum of the entropy. This was expressed in the variational problem of equation (II-16) as

$$\delta \int \sqrt{(dq^0/d\tau)^2}\ d\tau = 0. \tag{D-7}$$

Now consider a system which is describable by a Euclidean line element, equation (VI-30). For this system the h_{ij} are constants and the $h_{0\alpha}$ may be considered to be zero, since a suitable coordinate transformation may be found for which $h_{0\alpha}$ are zero. Hence equation (D-6) reduces to

$$(dq^0)^2 = \left(\frac{1}{h_{00}}\right) [c^2 dt^2 - h_{\alpha\beta} dq^\alpha\, dq^\beta]. \tag{D-8}$$

Since h_{00} is a constant for this system it may be factored out of the Euler equations which yield the trajectories satisfying the variational problem of equation (D-7); hence the problem is equivalent to finding geodesics in the space whose line element is

$$(ds)^2 = c^2 dt^2 - h_{\alpha\beta} dq^\alpha\, dq^\beta, \tag{D-9}$$

where $ds^2 = h_{00}(dq^0)^2$. When the $h_{\alpha\beta}$ are the coordinate coefficients $g_{\alpha\beta}$ of Euclidean space then equation (D-9) may be written as

$$(ds)^2 = c^2dt^2 - g_{\alpha\beta}\, dq^\alpha\, dq^\beta, \qquad \text{(D-10)}$$

which is the line element of the Minkowski space of special relativity. The Lorentz-Einstein transformation equations which leave equation (D-10) invariant are well known.

Since the Minkowski arc element ds was scaled to the differential change of entropy dq^0 by $ds = dq^0/\sqrt{h_{00}}$ a question of interpretation may arise. From equation (E-10) the proper time is seen to be

$$(ds)^2 = c^2(d\tau)^2 = c^2(dt)^2(1 - v^2/c^2)$$

where $v^2 = g_{\alpha\beta} \frac{dq^\alpha}{dt} \frac{dq^\beta}{dt}$. The entropy may then be seen to be associated with the proper time by

$$ds = c\; d\tau = dq^0/\sqrt{h_{00}}$$

or

$$dq^0 = c\sqrt{h_{00}}\;\; d\tau = c\; dt\sqrt{1 - v^2/c^2}\;\; .$$

The Minkowski length s is a measure of the "length" of the world line of a trajectory in Minkowski space. In Minkowski space, where there are no forces, a particle with an initial velocity will have a world line which increases its length indefinitely. If the entropy is proportional to the

Minkowski length then it too must increase indefinitely. Why should the entropy increase if there are no forces?

Consider a thermodynamic system consisting of an ideal gas at an initial pressure. If the pressure is completely removed the gas will expand freely and the entropy of the gas will increase indefinitely. Comparing this thermodynamic system to the free particle in Minkowski space it is consistent to expect the entropy of the free particle to increase with the Minkowski length.

Thus within this theory a Q-conservative system which is describable by a Euclidean line element must be described in Minkowski space when the system is in equilibrium at maximum entropy. This is not a new result because relativistic equations were arrived at in the section on isentropic systems but here the relationship between Minkowski space and the most general allowable space is more readily displayed and the transformation requirements are more easily seen.

Note that in Minkowski space $ds = 0$ for a light pulse. This corresponds to $dq^0 = 0$. It is an isentropic process and is consistent with the interpretation of light transmission as an isentropic process whose entropy is zero as required by the third law.

If a more general Riemannian space is considered the metric for an isolated system is no longer a Minkowski space. As an example consider the slightly more general space where $h_{\alpha\beta}$ and h_{00} are functions of the space coordinates but the $h_{0\alpha}$ are zero. Then equation (D-6) reduces to

$$(dq^0)^2 = \frac{c^2 dt^2}{h_{00}} - \frac{h_{\alpha\beta}}{h_{00}} dq^\alpha \, dq^\beta \tag{D-11}$$

again but now the remaining line element coefficients are functions of the space coordinates and the similarities between equation (D-11) and the Schwarzchild line element of general relativity may be seen. However, within this theory equation (D-6) represents only a portion of the allowed motion for though it is a general line element the variational problem of equation (D-7) may be used only for Q-conservative systems in equipoise. Other systems must use the line element and equations of Section VI.

A more general case than the two preceding examples would involve the mixed terms dt dq^0 . These terms are non-linear in the $h_{0\alpha}$ and are dependent upon the velocities dq^α/dt. When one or more of the $h_{0\alpha}$ are non-zero then one or more of these mixed terms appear in the line element for the Q-conservative system. An approximation of these terms may be made by expanding the square root factor. Since $h_{0\alpha} \neq 0$, then

$$\sqrt{\frac{c^2}{h_{00}} - \frac{h_{\gamma\delta}\dot{q}^\gamma \dot{q}^\delta}{h_{00}} + \left(\frac{h_{0\gamma}\dot{q}^\gamma}{h_{00}}\right)^2} = \frac{h_{0\gamma}\dot{q}^\gamma}{h_{00}} \sqrt{1 + \frac{h_{00}}{(h_{0\gamma}\dot{q}^\gamma)^2}(c^2 - h_{\gamma\delta}\dot{q}^\gamma \dot{q}^\delta)} \; . \tag{D-12}$$

$$\left(\frac{ds}{dt}\right)^2 \equiv c^2 - h_{\gamma\delta}\dot{q}^\gamma \, \dot{q}^\delta .$$

the right hand side of equation (D-12) may be written as

$$\frac{h_{0\gamma}\dot{q}^\gamma}{h_{00}}\sqrt{1 + \frac{h_{00}}{(h_{0\gamma}\dot{q}^\gamma)^2}\left(\frac{ds}{dt}\right)^2} \quad ,$$

then if $\frac{h_{00}}{(h_{0\gamma}\dot{q}^\gamma)^2}\left(\frac{ds}{dt}\right)^2 << 1$, this may be approximated by

$$\frac{h_{0\gamma}\dot{q}^\gamma}{h_{00}}\sqrt{1 + \frac{h_{00}}{(h_{0\gamma}\dot{q}^\gamma)^2}\left(\frac{ds}{dt}\right)^2} \cong \frac{h_{0\gamma}\dot{q}^\gamma}{h_{00}}\left[1 + \frac{h_{00}}{2(h_{0\gamma}\dot{q}^\gamma)^2}\left(\frac{ds}{dt}\right)^2\right]$$

Substituting this approximation into equation (D-6) and taking only the negative square root the line element becomes

$$(dq^0)^2 = \frac{c^2dt^2}{h_{00}} + \frac{2h_{0\gamma}}{h_{00}}\left\{-\frac{1}{2}\frac{h_{00}}{(h_{0\gamma}\dot{q}^\gamma)^2}\left(\frac{ds}{dt}\right)^2\right\}dtdq^\alpha - \frac{h_{\alpha\beta}}{h_{00}}dq^\alpha dq^\beta$$

$$= \frac{1}{h_{00}}\{c^2dt^2 - h_{\alpha\beta}dq^\alpha dq^\beta\} - \frac{h_{0\alpha}}{(h_{0\gamma}\dot{q}^\gamma)^2}\left(\frac{ds}{dt}\right)^2 dtdq^\alpha$$

or

$$\left(\frac{dq^0}{dt}\right) = \frac{1}{h_{00}}\left(\frac{ds}{dt}\right)^2 \left[1 - \frac{h_{00}(h_{0\gamma}\dot{q}^\gamma)}{(h_{0\gamma}\dot{q}^\gamma)^2}\right]$$

so that

$$(dq^0)^2 \cong \frac{(ds)^2}{h_{00}}\left[1 - \frac{h_{00}}{(h_{0\gamma}\dot{q}^\gamma)^2}\right] . \qquad \text{(D-13)}$$

By comparing equation (D-13) with equation (D-11) it is possible to see how the mixed terms in the line element (D-6) affect the isolated line element (D-11).

This Appendix does not attempt to answer all questions concerning transformation requirements, and is presented with the hope that the reader may see that this theory does not require a rework of previous accomplishments of physics, but rather includes them while providing a framework which may include others as well.

Recall that in Chapter V the arc element was only three-dimensional. The three-dimensional arc element of that chapter describes an isentropic, Q-conservative system. The system in this Appendix in only Q-conservative and thus is four-dimensional since the mechanical entropy is not required to remain constant.

APPENDIX E

EXPANSION OF PLANETARY ORBITS

There is considerable demand on any newly proposed theory to predict some new phenomenon. This demand may stem from the desire to create a crucial experiment to help determine the theory's validity, or it may arise from the desire to demonstrate an expansion of scope, or increase of generality, over prior theories.

During this investigation the time was primarily spent in the basic formulation of the dynamic laws and investigating the possible consistency. However, a recent article in Scientific American[7] discussed the necessity of a changing gravitational constant to account for an observed expansion of the moon's orbit that was not attributable to known effects. It then became interesting to consider what, if anything, the proposed theory would predict concerning changing orbit size.

Recalling the classical problem in electrodynamics of calculating the time it would take for an electron to spiral down and into a proton, it is natural to assume that if energy is radiated away from this system that the orbit size should decrease. The electron-proton problem immediately

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[7] Van Flandern, T.C., "Is Gravity Getting Weaker?" Scientic American, v. 234, #2, p. 44-52, February 1976.

brings decreasing planetary orbits to mind. This notion involves the idea of energy being radiated away. Figure E1 illustrates how radiation of energy results in a decrease of the orbit for the electron-proton problem.

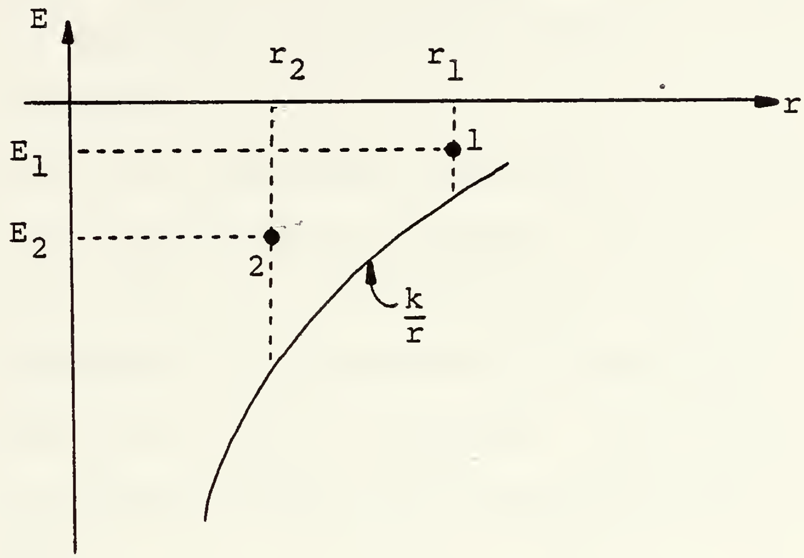

Figure E1. The decrease in energy $E_2 - E_1$ results in the decrease in radius $r_2 - r_1$.

Now a Q-conservative system does not radiate any energy because $\bar{d}Q = 0$. If the simplified two-body system is considered to be the entire universe it must of necessity be Q-conservative. Recalling that one result of the second law, seen through Clausius' theorem, equation (II-13), was that the entropy of an isolated system never decreases and remains constant only for reversible processes, in particular, recall equation (II-15)

$$\int_I \frac{\bar{d}Q}{\phi} - \int_R \frac{\bar{d}Q}{\phi} \equiv S(B) - S(A). \qquad \text{(II-15)}$$

Now if the motion of the system is considered to be a reversible process then, since it is Q-conservative, it must also be isentropic and should be describable by isentropic equations. If the system is not describable by isentropic equations but is Q-conservative then the process must be an irreversible one and the entropy of the system must be increasing.

In Section V the isentropic equations of motion yield the relativistic Hamiltonian, equation (V-12) as a constant of the motion. Then since the Hamiltonian and the entropy are both constants of the motion for an isentropic system they can differ by at most a constant. Then taking the Hamiltonian and the entropy to be the same value, equation (V-12) becomes

$$S_0 = mc^2[\frac{1}{\sqrt{1 - v^2/c^2}} - 1] + V(q^1,q^2,q^3). \qquad \text{(E-1)}$$

If $V(q^1,q^2,q^3)$ is taken as the Newtonian gravitation potential in spherical coordinates, equation (E-1) becomes

$$S_0 = mc^2 [\frac{1}{\sqrt{1 - v^2/c^2}} - 1] + \frac{k}{r} \qquad \text{(E-2)}$$

where $k = -GMm$. Equation (E-2) is the energy of the system in special relativity and does not quite describe planetary motion. Special relativity accounts for only one-sixth of the Perihelion motion which is not already explained by

planet perturbations. Then if the system is Q-conservative and the isentropic equation (E-2) does not describe the motion the process must be irreversible and the entropy must be increasing.

If the entropy is increasing very slowly equation (E-2) gives a close approximation to the motion. However, using Figure E2 and equation (E-2) the situation may be seen to be the reverse of the electron-proton problem.

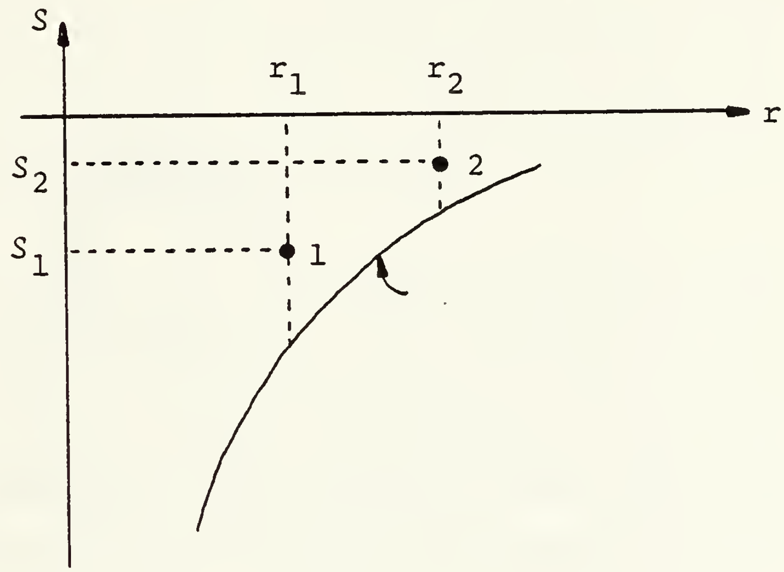

Figure E2. The increase in entropy $S_2 - S_1$ results in the increase in radius $r_2 - r_1$.

In the low velocity limit equation (E-2) becomes the Newtonian energy of the system

$$S_0 = E = \frac{1}{2} mv^2 + k/r \qquad \text{(E-3)}$$

and an increase of entropy in the low velocity limit corresponds to an increase in system energy which implies work, $đQ \neq 0$, has been done on the system. How can an increase of entropy be consistent with a Q-conservative system where $đQ = 0$?

Recall that, from Section V,

$$đQ = m\dot{q}^{\alpha}dq^{\alpha} - \sqrt{1 - v^2/c^2}\; F_{\alpha}\, dq^{\alpha}$$

while

$$dS = \frac{m\dot{q}^{\alpha}dq^{\alpha}}{\sqrt{1 - v^2/c^2}} - F_{\alpha}\, dq^{\alpha}$$

since $dS = \dfrac{đQ}{\sqrt{1 - v^2/c^2}}$. In the low velocity limit $dS \rightarrow đQ$. If $dS = đQ$ then $đQ$ would be a perfect differential and the integrating factor would not have been needed. It is because of the existence of the integrating factor that the difference between dS and $đQ$ exists, and it is this difference which allows a system to remain Q-conservative, $đQ = 0$, while the entropy increases. The relationship between $\bar{v}$ and $\bar{r}$ is governed by the conservation of angular momentum. This relationship determines how total entropy is split between kinetic and potential entropy, just as classically the angular momentum in the central force problem determines the split between the kinetic and potential energy.

Thus, even though this increase in orbit size must be very slow if it is to correspond to experimental reality, it is a reasonable qualitative prediction of the theory provided the orbit motion is considered to be an irreversible process of a Q-conservative system.

APPENDIX F

OTHER METHODS OF DETERMINING THE COEFFICIENTS

The element on classical thermodynamics which makes the theoretical logic consistent with experimental results is the equation of state, while in Newtonian mechanics it is the force law. In general in the application of this dynamic theory, in the non-isentropic case, correspondence between theory and reality has to be made by the choice of the metric coefficients. Then to apply the theory to any physical situation it is necessary to determine the applicable coefficients. But how may they be determined for the different situations?

In order to investigate the possibility of methods of determining the coefficients other than the assumption of equations involving Einstein's tensor equation, for instance consider the four-dimensional line element given by equation (VI-30)

$$(ds)^2 = h_{ij}\, dq^i\, dq^j \ ; \quad i,j = 0,1,2,3, \qquad \text{(F-1)}$$

where $q^0 = S/F_0$ is the entropy. There is more than one method of determining equations of motion for this system.

A. ARC LENGTH AS THE PARAMETER

Therefore suppose, as the first approach, the parameter is chosen to be the arc length by the choice

$$ds \equiv c\,dt. \tag{F-2}$$

Then equation (F-1) may be written

$$c^2 \equiv \left(\frac{ds}{dt}\right)^2 = h_{0}(\dot{q}^0)^2 + 2h_{0\alpha}(\dot{q}^0)\dot{q}^\alpha + h_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta \tag{F-3}$$

where $\dot{q}^i = dq^i/dt$. Then define

$$\tilde{L} \equiv \frac{1}{2}\,mc^2 = \frac{mh_{00}}{2}(\dot{q}^0)^2 + mh_{0\alpha}\dot{q}^0\dot{q}^\alpha + \frac{mh_{\alpha\beta}}{2}\,\dot{q}^\alpha\dot{q}^\beta \tag{F-4}$$

where m is a constant. Now define

$$\tilde{\Phi} \equiv -\frac{1}{2}\,h_{00}(\dot{q}^0)^2 \;, \tag{F-5}$$

$$\frac{\tilde{A}_\alpha}{c} \equiv h_{0\alpha}(\dot{q}^0) \;, \tag{F-6}$$

and

$$\tilde{K} \equiv \frac{m}{2}\,h_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta \;, \tag{F-7}$$

so that equation (F-4) may be written as

$$\tilde{L} = -\,m\tilde{\Phi} + \frac{mA_\alpha}{c}\,\dot{q}^\alpha + \tilde{K} \;. \tag{F-8}$$

The geodesics are then given by the four equations

$$\frac{d}{dt}[\dot{\partial}_i \tilde{L}] - \partial_i \tilde{L} = 0; \quad i = 0,1,2,3, \tag{F-9}$$

where

$$\dot{\partial}_i \equiv \frac{\partial}{\partial \dot{q}^i} .$$

The equation for i = 0 is then

$$\frac{d}{dt}[-m\, \dot{\partial}_0 \tilde{\Phi} + \frac{m}{c}\, \dot{\partial}_0 (\tilde{A}_\alpha \dot{q}^\alpha)] - \partial_0 \tilde{L} = 0, \tag{F-10}$$

while the three spacial equations are

$$\frac{d}{dt}[\dot{\partial}_\alpha \tilde{L}] - \partial_\alpha \tilde{L} = 0; \quad \alpha = 1,2,3. \tag{F-11}$$

By defining

$$\tilde{V} \equiv m\tilde{\Phi} - \frac{m\tilde{A}_\alpha}{c}\, \dot{q}^\alpha$$

equation (F-8) can be rewritten as

$$\tilde{L} = \tilde{K} - \tilde{V}. \tag{F-12}$$

Equation (F-11) becomes

$$\frac{d}{dt}[\dot{\partial}_\alpha \tilde{K}] - \partial_\alpha \tilde{K} = \frac{d}{dt}[\dot{\partial}_\alpha \tilde{V}] - \partial_\alpha \tilde{V}. \tag{F-13}$$

Note that here the metric coefficients are predetermined by some physical law, equation of state, or empirically. However when the h_{ij} are known then equation (F-10) may be solved for $\dot{q}^0$ and equations (F-13) yield the $\dot{q}^\alpha$.

Define the right hand side of equation (F-13) as the force

$$\tilde{f}_\alpha \equiv \frac{d}{dt}[\dot{\partial} \quad \tilde{V}] - \partial_\alpha \tilde{V}. \tag{F-14}$$

Note that

$$\dot{\partial}_\alpha \tilde{V} = - \frac{m}{c} \tilde{A}_\alpha$$

then

$$\frac{d}{dt}[\dot{\partial}_\alpha \tilde{V}] = - \frac{m}{c}\{ \partial\beta \tilde{A}_\alpha \dot{q}^\beta + \partial_0 \tilde{A}_\alpha \dot{q}^0\}$$

and

$$\partial_\alpha \tilde{V} = m \partial_\alpha \tilde{\Phi} - \frac{m}{c} \partial_\alpha \tilde{A}_\beta \dot{q}^\beta.$$

Substituting these expressions into the force (F-14) yields

$$\tilde{f}_\alpha = - \frac{m}{c}\partial_\beta\tilde{A}_\alpha\dot{q}^\beta - \frac{m}{c}\partial_0\tilde{A}_\alpha\dot{q}^0 - m\partial_\alpha\tilde{\Phi} + \frac{m}{c}\partial_\alpha\tilde{A}_\beta\dot{q}^\beta$$

or

$$\tilde{f}_\alpha = -m\partial_\alpha\tilde{\Phi} - \frac{m}{c}\partial_0\tilde{A}_\alpha\dot{q}^0 + \frac{m}{c}\{\partial_\alpha\tilde{A}_\beta\dot{q}^\beta - \partial_\beta\tilde{A}_\alpha\dot{q}^\beta\}. \tag{F-15}$$

Define

$$\tilde{E}_\alpha \equiv - \partial_\alpha\tilde{\Phi} - \frac{1}{c}\partial_0\tilde{A}_\alpha\dot{q}^0 \tag{F-16}$$

and

$$\tilde{\bar{B}} \equiv \nabla \times \tilde{\bar{A}} , \tag{F-17}$$

so that the force may be written as

$$\tilde{f}_\alpha = m\,\tilde{E}_\alpha + \frac{m}{c}(\bar{v} \times \tilde{\bar{B}})_\alpha$$

where

$$v_\alpha \equiv \dot{q}^\alpha$$

or as a vectoral equation

$$\tilde{\bar{f}} = m\tilde{\bar{E}} + \frac{m}{c}(\bar{v} \times \tilde{\bar{B}}) . \tag{F-18}$$

It has been shown here that forces resulting from the four metric coefficients $h_{0\alpha}$ and h_{00} can be brought into the form of a Lorentz force and that this way it is shown how, for

electromagnetic forces, the electromagnetic forces, the electromagnetic potentials are connected with the metric.

There are two questions which may be posed here. Suppose the potentials defined here are the electromagnetic potentials. Then the electrostatic potential would be a function of the square of the rate of entropy change while the components of the vector potential would be proportional to the rate of entropy change. What would be the physical significance of this relationship? If the $h_{\alpha\beta}$ are to correspond to the isentropic metric elements there may be a force involved with them (see equations (B-6) and (V-27)). What then is the physical relationship between these forces and the Lorentz-like forces of equation (F-18)? If the Lorentz-like forces are in fact the electromagnetic forces, what type of force must the others be?

B. ENTROPY AS THE PARAMETER

A second approach might be to choose one coordinate as the parameter, such as choosing

$$d\dot{q} \equiv c\, d\tau. \tag{F-19}$$

For this choice of parameter the line element becomes

$$\left(\frac{ds}{d\tau}\right)^2 = c^2\, h_{00} + 2c\, h_{0\alpha} q'^{\alpha} + h_{\alpha\beta}\, q'^{\alpha}\, q'^{\beta}\ ,$$

where

$$q'^{\alpha} \equiv dq^{\alpha}/d\tau .$$

The geodesics are then found as the solution to the variational problem

$$\delta \int \sqrt{(ds/d\tau)^2}\, d\tau = 0 \tag{F-20}$$

or defining

$$\underset{\sim}{L} \equiv \frac{m}{2}\left(\frac{ds}{d\tau}\right)^2 = \frac{mc^2}{2} h_{00} + mc\, h_{0\alpha}\, q'^{\alpha} + \frac{m}{2}\, h_{\alpha\beta}\, q'^{\alpha}\, q'^{\beta} \tag{F-21}$$

and

$$\underset{\sim}{K} \equiv \frac{m}{2}\, h_{\alpha\beta} q'^{\alpha}\, q'^{\beta}, \tag{F-22}$$

with the additional definitions

$$\underset{\sim}{\Phi} \equiv -\frac{1}{2}\, c^2\, h_{00}, \tag{F-23}$$

$$\frac{\underset{\sim}{A}_{\alpha}}{c} \equiv c\, h_{0\alpha}, \tag{F-24}$$

$$\underset{\sim}{V} \equiv m\, \underset{\sim}{\Phi} - \frac{m}{c}\, \underset{\sim}{A}_{\alpha}\, q'^{\alpha}, \tag{F-25}$$

then

$$\underset{\sim}{L} = \frac{m}{2}\left(\frac{ds}{d\tau}\right)^2 = \underset{\sim}{K} - \underset{\sim}{V} \tag{F-26}$$

The integrand in the variational problem, equation (F-20) is a function of τ, q^α and q'^α, therefore there are only three equations of motion. To obtain these equations consider

$$F \equiv \sqrt{h_{ij}\, q'^i\, q'^j} = \sqrt{\underset{\sim}{L}} = \sqrt{\underset{\sim}{K} - \underset{\sim}{V}}\ ,$$

then

$$\partial'_\alpha F = \frac{1}{2}\underset{\sim}{L}^{-\frac{1}{2}}(\partial'_\alpha \underset{\sim}{K} - \partial'_\alpha \underset{\sim}{V}),$$

and

$$\partial_\alpha F = \frac{1}{2}\underset{\sim}{L}^{-\frac{1}{2}}(\partial_\alpha \underset{\sim}{K} - \partial_\alpha \underset{\sim}{V}).$$

Euler's equations then become

$$\frac{d}{d\tau}[\underset{\sim}{L}^{-\frac{1}{2}}(\partial'_\alpha \underset{\sim}{K} - \partial'_\alpha \underset{\sim}{V})] - \underset{\sim}{L}^{-\frac{1}{2}}(\partial_\alpha \underset{\sim}{K} - \partial_\alpha \underset{\sim}{V}) = 0. \tag{F-27}$$

Carrying out the differentiation of one of the terms of the product and rearranging terms leads to

$$\frac{d}{d\tau}[\partial'_\alpha \underset{\sim}{K}] - \partial_\alpha \underset{\sim}{K} = \frac{d}{d\tau}[\partial'_\alpha \underset{\sim}{V}] - \partial_\alpha \underset{\sim}{V} + f_{D\alpha}, \tag{F-28}$$

where

$$f_{D\alpha} \equiv \frac{1}{2} L^{-1}(\partial'_\alpha \underset{\sim}{K} - \partial'_\alpha \underset{\sim}{V})[\partial'_\beta \underset{\sim}{L}\, q''^{\beta} + \partial_\beta \underset{\sim}{L}\, q'^{\beta} + \partial_\tau \underset{\sim}{L}]. \tag{F-29}$$

But from equations (F-23), (F-24), and (F-25)

$$\partial'_\alpha \underset{\sim}{V} = -\frac{m}{c} \underset{\sim}{A}_\alpha$$

and

$$\partial_\alpha \underset{\sim}{V} = m\partial_\alpha \underset{\sim}{\Phi} - \frac{m}{c} \underset{\sim}{A}_\beta\, q'^{\beta},$$

so that

$$\frac{d}{d\tau}[\partial'_\alpha \underset{\sim}{V}] = -\frac{m}{c}[\partial_\beta \underset{\sim}{A}_\alpha\, q'^{\beta} + \partial\tau\, \underset{\sim}{A}_\alpha],$$

where

$$\partial_\tau \equiv \frac{\partial}{\partial\tau}\ .$$

Then equation (F-28) may be written as

$$\frac{d}{d\tau}[\partial'_\alpha K] - \partial_\alpha \underset{\sim}{K} = -m\partial_\alpha \underset{\sim}{\Phi} - \frac{m}{c}\partial\tau \underset{\sim}{A}_\alpha + \frac{m}{c}[\partial_\alpha \underset{\sim}{A}_\beta q'^{\beta} - \partial_\beta \underset{\sim}{A}_\alpha q'^{\beta}]. \tag{F-30}$$

The definitions

$$\underset{\sim}{E}_{\alpha} \equiv - \partial_{\alpha} \underset{\sim}{\Phi} - \frac{1}{c} \partial_{\tau} \underset{\sim}{A}_{\alpha} \tag{F-31}$$

and

$$\underset{\sim}{\overline{B}} \equiv \nabla \times \underset{\sim}{\overline{A}} \tag{F-32}$$

may be made so that equation (F-30) can be expressed as

$$\frac{d}{d\tau}[\partial'_{\alpha} \underset{\sim}{K}] - \partial_{\alpha} \underset{\sim}{K} = \underset{\sim}{f}_{\alpha} + f_{D_{\alpha}} \tag{F-33}$$

where

$$\underset{\sim}{\overline{f}} = m\underset{\sim}{\overline{E}} + \frac{m}{c}(\underset{\sim}{\overline{v}} \times \underset{\sim}{\overline{B}}), \tag{F-34}$$

with

$$\underset{\sim}{\overline{v}}_{\alpha} \equiv q'^{\alpha}.$$

These two approaches may be applied to any four-dimensional metric, the difference being that in the first case, where the arc length was considered as the parameter, four equations of motion were obtained which would describe the motion as geodesics. In the second case there are only three equations of motion from the variational principle. Given the coefficients h_{ij} which correspond to the physical situation either procedure might be used to obtain the motion.

The coefficients $h_{0\alpha}$ and h_{00} are the coefficients which describe the deviation from an isentropic system and the potentials defined from these coefficients lead to a force ($\tilde{\bar{f}}$ or $\underset{\sim}{\bar{f}}$) similar in form to the Lorentz forces in electromagnetism. This suggests that the description of "radiation" of entropy is provided by these coefficients.

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